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A Finite Volume Method for the Relativistic Burgers Equation on a FLRW Background Spacetime

Tuba Ceylan[1], Philippe G. LeFloch[2,*] and Baver Okutmustur[1]

[1] *Department of Mathematics, Middle East Technical University (METU), 06800 Ankara, Turkey.*
[2] *Laboratoire Jacques-Louis Lions & Centre National de la Recherche Scientifique, Université Pierre et Marie Curie (Paris 6), 4 Place Jussieu, 75252 Paris, France.*

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Abstract. A relativistic generalization of the inviscid Burgers equation was introduced by LeFloch and co-authors and was recently investigated numerically on a Schwarzschild background. We extend this analysis to a Friedmann-Lemaître-Robertson-Walker (FLRW) background, which is more challenging due to the existence of time-dependent, spatially homogeneous solutions. We present a derivation of the model of interest and we study its basic properties, including the class of spatially homogeneous solutions. Then, we design a second-order accurate scheme based on the finite volume methodology, which provides us with a tool for investigating the properties of solutions. Computational experiments demonstrate the efficiency of the proposed scheme for numerically capturing weak solutions.

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1 Introduction

Aim of this paper

The inviscid Burgers equation is an important model in computational fluid dynamics and provides the simplest (yet challenging) example of a nonlinear hyperbolic conservation law. Recently, a relativistic generalization of the standard Burgers equation was introduced on curved spacetimes and studied by LeFloch and collaborators [1–4,9–12]. This

*Corresponding author. *Email addresses:* `ceylanntuba@gmail.com` (T. Ceylan), `contact@philippelefloch.org` (P. G. LeFloch), `baver@metu.edu.tr` (B. Okutmustur)

relativistic Burgers equation, as it is now called, takes into account geometrical effects and satisfies the Lorentz invariance property, also enjoyed by the Euler equations of relativistic compressible fluids. In LeFloch, Makhlof, and Okutmustur [11], this model was studied on a Schwarzschild black hole background and a numerical scheme was designed from the finite volume methodology which allowed one to capture weak solutions containing shock waves. The standard maximum principle and the total variation diminishing (TVD) principle are lacking for Burgers-type equations on a curved geometry, so that it necessary to revisit standard approaches if stable approximation schemes are sought for. Our main objective in the present paper is to extend the results in [11] and design an accurate and robust numerical scheme when the model of interest is formulated on a cosmological background and, especially, on a Friedmann-Lemaître-Robertson-Walker (FLRW) geometry. This latter spacetime is a solution to Einstein's field equations and is particularly relevant in cosmology [7].

The relativistic Burgers equations on a curved background

Our model is derived from the full relativistic Euler equations posed on a smooth, time-oriented Lorentzian manifold (M,g), that is,

$$\nabla_\alpha T^{\alpha\beta}=0, \qquad T^{\alpha\beta}=(\rho c^2+p)u^\alpha u^\beta+pg^{\alpha\beta},$$

where $T^{\alpha\beta}$ is the energy-momentum tensor of a perfect fluid. Here, $\rho\geq 0$ denotes the mass-energy density of the fluid, while the future-oriented, unit timelike vector field $u=(u^\alpha)$ represents the fluid velocity, normalized to be unit: $g_{\alpha\beta}u^\alpha u^\beta=-1$.

As usual, the Euler equations are supplemented with an equation of state for the pressure $p=p(\rho)$. In the present work, we assume that the fluid is pressureless, that is, $p\equiv 0$, so that the Euler system takes the simpler form

$$\nabla_\alpha\big(\rho u^\alpha u^\beta\big)=0. \tag{1.1}$$

Provided ρ,u are sufficiently regular, we can write

$$\rho\nabla_\alpha u^\alpha u^\beta+\rho u^\alpha\nabla_\alpha u^\beta+u^\alpha u^\beta\nabla_\alpha\rho=0.$$

By contracting this equation with the covector u_β and by observing[†] that $g_{\alpha\beta}u^\beta\nabla_\gamma u^\alpha=0$, we get $u^\alpha\nabla_\alpha\rho=-\rho\nabla_\alpha u^\alpha$. In turn, from (1.1) it follows that

$$\rho u^\beta\nabla_\alpha u^\alpha+\rho u^\alpha\nabla_\alpha u^\beta-\rho u^\beta\nabla_\alpha u^\alpha=0.$$

Finally, provided $\rho>0$, it follows that

$$u^\alpha\nabla_\alpha u^\beta=0, \tag{1.2}$$

which we refer to as the *relativistic Burgers equation* (expressed here in a non-divergence form) on a curved background (M,g). Its unknown is the vector field (u^β) satisfying the normalization $g_{\alpha\beta}u^\alpha u^\beta$.

[†] u is orthogonal to ∇u, as is easily checked by differentiating the identity $g_{\alpha\beta}u^\alpha u^\beta=-1$ stating that u is unit vector and by recalling that ∇ is the Levi-Civita connection associated with the metric $g_{\alpha\beta}$ so that $\nabla_\gamma g_{\alpha\beta}=0$.