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# Preconditioning of a Coupled Cahn-Hilliard Navier-Stokes System

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**Abstract.** Recently, Garcke et al. [H. Garcke, M. Hinze, C. Kahle, Appl. Numer. Math. 99 (2016), 151–171)] developed a consistent discretization scheme for a thermodynamically consistent diffuse interface model for incompressible two-phase flows with different densities [H. Abels, H. Garcke, G. Grün, Math. Models Methods Appl. Sci. 22(3) (2012)]. At the heart of this method lies the solution of large and sparse linear systems that arise in a semismooth Newton method.

In this work we propose the use of preconditioned Krylov subspace solvers using effective Schur complement approximations. Numerical results illustrate the efficiency of our approach. In particular, our preconditioner is shown to be robust with respect to parameter changes.

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## 1 Introduction

In recent years numerical simulation of two-phase flow and multiphase flow has attained growing attention both from the modeling point of view and the development of schemes for the numerical realization. This especially holds for so called diffuse interface or phase-field models, where the distribution of the two phases, i.e. the regions where the two phases are located, is encoded in a smooth indicator function, called phase field. In

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this work we consider the particular problem of solving linear systems arising during Newton's method for the solution of nonlinear equations appearing in an energy stable time stepping scheme for a thermodynamically consistent model for two-phase flow.

We use the thermodynamically consistent model proposed in [6] and the numerical scheme presented in [25]. The model is based on a phase field representation of the two-phase fluid structure and couples the Navier-Stokes equations to the Cahn-Hilliard equations in a thermodynamically consistent way, i.e., an energy equality holds.

Before we state the model and the scheme below in Section 2, let us briefly comment on the development of models for two-phase flows and especially on numerical schemes for these. The first model for two-phase flow using a phase field representation is the model 'H' proposed in [33]. It couples the Navier-Stokes equation and the Cahn-Hilliard equation [19] in a thermodynamically consistent way. It is only valid for fluids of equal density. Since then several attempts have been made for a generalization to fluids of different densities, see e.g. [15,43,57]. The first thermodynamically consistent model, that contains the same coupling between the Navier-Stokes and the Cahn-Hilliard model is proposed in [6]. It contains an additional flux in the Navier-Stokes equation that appears due to the phase field representation of the two phases and due to the different densities of the involved fluids. Especially for equal density fluids, this flux vanishes and in this case the model equals the model 'H'.

In the few years since the development of this model, several groups proposed schemes for its numerical realization and we refer here only to [3, 7, 24–26, 28, 29, 36]. The first two schemes decouple the Navier-Stokes and the Cahn-Hilliard equation. This means that on each time instance, the Cahn-Hilliard equation can be solved in advance and the solution can be used to solve the Navier-Stokes equation afterwards. The systems are decoupled by using an augmented velocity field in the Cahn-Hilliard equation, that then only depends on the velocity field from the old time instance and thus can be solved before turning to the Navier-Stokes equation. This procedure adds some diffusion outside of the interface to the Cahn-Hilliard equation. In this way, the defects from using the old velocity field can be balanced with numerical dissipation that arises due to time discretization to obtain an energy stable scheme.

The other papers propose fully coupled schemes and especially in [7] benefits from using a fully coupled scheme are investigated. Aland addressed integration strategies for diffuse interface models for two-phase flows. The fully implicit scheme turned out to be stable for arbitrarily large time steps. Also, the fully coupled system needs by far the least number of inner fix-point iterations. At the heart of the numerical realization of such fully coupled schemes large and sparse linear systems have to be solved, that have a block structure, namely

$$\mathcal{A}=\left(\begin{array}{c|c} \boldsymbol{A}_{\mathrm{NS}} & \boldsymbol{C}_I \\ \hline \boldsymbol{C}_T & \boldsymbol{A}_{\mathrm{CH}} \end{array}\right),$$

where $\boldsymbol{A}_{\mathrm{NS}}$ and $\boldsymbol{A}_{\mathrm{CH}}$ are themselves block matrices arising from the discretization of the (linearized) Navier-Stokes equation ($\boldsymbol{A}_{NS}$) and the (linearized) Cahn-Hilliard equation