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Optimal Convergence Analysis of a Mixed Finite Element Method for Fourth-Order Elliptic Problems

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Abstract. A Ciarlet-Raviart type mixed finite element approximation is constructed and analyzed for a class of fourth-order elliptic problems arising from solving various gradient systems. Optimal error estimates are obtained, using a super-closeness relation between the finite element solution and the Ritz projection of the PDE solution. Numerical results agree with the theoretical analysis.

AMS subject classifications: 65N12, 65N30, 35J30

Key words: Fourth-order elliptic problems, mixed finite element, optimal convergence.

1 Introduction

In this paper, we study a mixed finite element method for the problem:

$$\Delta^2 u - \nabla\cdot(a\nabla u) + bu = f \qquad \text{in } \Omega, \tag{1.1}$$

with boundary condition

$$\frac{\partial u}{\partial \mathbf{n}} = \frac{\partial \Delta u}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega, \tag{1.2}$$

where $\mathbf{n}$ denotes the unit outward normal vector along the boundary of the domain $\Omega \subset \mathbb{R}^d$, and a, b, f are given scalar-valued functions. Note that when $a=b\equiv 0$, Problem (1.1)-(1.2) can be decoupled into two Poisson's equations. This trivial case will be ruled out as

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we impose the following assumptions:

$$0 \le a(x) \le \bar{a}, \quad 0 < \underline{b} \le b(x) \le \bar{b}, \qquad \forall x \in \Omega. \tag{1.3}$$

Problem (1.1)-(1.2) originates, though with much simplification, from a wide class of important physical models, such as the phase field crystal (PFC) equation/modified phase field crystal (MPFC) equation [3,4,28,36,38], epitaxial thin film growth models [8,9, 11,12,34,35], non-local Cahn-Hilliard-type models [24,25], the CHHS and related models [10,15,16,19,26,37]. A main feature of our work is to focus on the Neumann and the no-slip boundary conditions (1.2), which can be important in many physical applications. Note that the boundary condition (1.2) differs from the familiar

$$\begin{aligned}
\text{Dirichlet boundary condition: } & u = \frac{\partial u}{\partial \mathbf{n}} = 0 && \text{on } \partial\Omega, \\
\text{Neumann boundary condition: } & \Delta u = \frac{\partial \Delta u}{\partial \mathbf{n}} = 0 && \text{on } \partial\Omega, \\
\text{Navier boundary condition: } & u = \Delta u = 0 && \text{on } \partial\Omega.
\end{aligned}$$

One may view (1.2) as a Neumann-Navier type condition [20]. For fourth-order elliptic equations with boundary condition (1.2), well-posedness and other properties of the exact solution are non-trivial and remain largely unknown. Interested readers may refer to Chapter 2 of the monograph [21] for more details. In this paper, we shall focus on the numerical approximation to Problem (1.1)-(1.2) only when it admits a solution. In other words, we simply assume for certain combinations of a, b and f: *There exists a* $u \in H^2(\Omega)$ *that solves Problem* (1.1)-(1.2). In many cases, the existence of the solution is indeed guaranteed by the physical background of the problem. This assumption immediately implies the existence of solution to a mixed formulation for problem (1.1)-(1.2), derived by introducing a new variable $w = -\Delta u$. That is: *Find* $u \in H^1(\Omega)$, $w \in \overline{H^1}(\Omega)$ *satisfying*

$$\begin{cases} (w,v) - (\nabla u, \nabla v) = 0, & \forall v \in \overline{H^1}(\Omega), \\ (\nabla w, \nabla \psi) + (a \nabla u, \nabla \psi) + (bu, \psi) = (f, \psi), & \forall \psi \in H^1(\Omega), \end{cases} \tag{1.4}$$

where $\overline{H^1}(\Omega)$ is the mean-value free subspace of $H^1(\Omega)$ and $(\cdot,\cdot)$ is the L^2 inner product over Ω. Such kind of mixed formulation, i.e., where the dual variable is Δu, is known as the Ciarlet-Raviart mixed formulation [14] in the literature. Assuming the existence of the solution, then the uniqueness of the solution to (1.4) is obvious by noticing that when $f = 0$ the system only admits a trivial solution.

The main purpose of this paper is to study an equi-order finite element discretization of (1.4). Let $S_h \subset H^1(\Omega)$ be a finite dimensional space and $\overline{S}_h \subset \overline{H^1}(\Omega)$ be the mean-value free subspace of S_h. The discrete weak formulation can thus be written as: *Find* $u_h \in S_h$, $w_h \in \overline{S}_h$ *satisfying*

$$\begin{cases} (w_h, v_h) - (\nabla u_h, \nabla v_h) = 0, & \forall v_h \in \overline{S}_h, \\ (\nabla w_h, \nabla \psi_h) + (a \nabla u_h, \nabla \psi_h) + (b u_h, \psi_h) = (f, \psi_h), & \forall \psi_h \in S_h. \end{cases} \tag{1.5}$$