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A Fast Finite Difference Method for Tempered Fractional Diffusion Equations

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Abstract. Using the idea of weighted and shifted differences, we propose a novel finite difference formula with second-order accuracy for the tempered fractional derivatives. For tempered fractional diffusion equations, the proposed finite difference formula yields an unconditionally stable scheme when an implicit Euler method is used. For the numerical simulation and as an application, we take the CGMYe model as an example. The numerical experiments show that second-order accuracy is achieved for both European and American options.

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Key words: Tempered fractional derivatives, fractional differential equations, method of characteristics, CGMYe model.

1 Introduction

Fractional derivatives were invented by Leibnitz soon after the integer order derivatives, but have been playing more and more important roles in recent decades. They are now applied to model a wide variety of problems, including in mechanics (theory of viscoelasticity and viscoplasticity), (bio-)chemistry (modeling of polymers and proteins), electrical engineering (transmission of ultrasound waves), and medicine (modeling of human tissue under mechanical loads), etc.

The finite difference method is the most commonly used method in solving the fractional partial differential equation (FPDE) problems. The tempered Riemann–Liouville fractional derivative can be discretized by the standard Grünwald–Letnikov formula [20]

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with only the first-order accuracy, but the difference scheme based on the Grünwald–Letnikov formula for time dependent problems is unstable [18]. To overcome this problem, Sabzikar *et al.* first proposed the shifted Grünwald–Letnikov formula in [21] to approximate the tempered fractional advection-dispersion flow equations, which is stable with first-order accuracy in space. In the present work, we propose a stable and more flexible approach to approximate the tempered Riemann–Liouville fractional derivatives via the weighted average of distinct shifted Grünwald–Letnikov formulas, and to achieve second-order accuracy. The idea of this method is under inspiration of the method used by Tian *et al.* in [27].

For the numerical experiments and as an application of the proposed method, we shall consider the CGMYe model for option pricing problem. This model is proposed by Carr, Geman, Madan, and Yor [5], and involves four parameters C, G, M, and Y with one extra parameter η standing for the volatility, hence the name "CGMYe model". In finance, fractional operators are used to develop mathematical models that can describe the dynamics of asset price more precisely compared to the classical diffusion operators. By adopting a Lévy process extending Brownian motion for the description of the price, large price changes, due to sudden exogenous events on information and some systematic empirical biases with respect to the traditional Black–Scholes model [4], can be explained and described properly [1]. The CGMYe process model is one kind of tempered stable process models. Comparing with the other jump models, the CGMYe model can adequately describe the empirical features of asset returns and at the same time provide a reasonable fit to the implied volatility surfaces observed in option markets. Therefore the CGMYe model becomes popular in practice, while the fast and efficient computation of this model is still a challenging problem [2, 10, 11, 17].

It is known that the CGMYe model obeys a tempered FPDE [13, 14]. In the numerical experiments, we apply the proposed finite difference formulas to the CGMYe model equation. For the advection term in this equation, we adopt the second-order upwind scheme and the method of characteristic lines. The results of the numerical experiments show that both methods converge faster and cost less CPU time than the traditional methods for both European and American option pricing problems. Besides the univariate CGMYe model, we also take consideration of the two-dimensional pricing problems, *i.e.*, to solve the 2-D FPDE of the CGMYe model. Here, we use the alternating directional implicit (ADI) method in the implementation, and the second-order convergence is also observed in the numerical simulations.

The rest of this paper is organized as follows. In Section 2, we propose the difference operators to approximate the tempered Riemann–Liouville fractional derivatives with second-order truncation error, and use the upwind scheme and the method of characteristic lines for the discretization of the FPDE. We then prove the consistency and the stability of the second-order approximation in Section 3 and analyze the properties of the coefficient matrix in Section 4. In Section 5, we extend the one dimensional problem to a two dimensional problem, namely, the multi-asset CGMYe model, and use the ADI method to discretize the 2-D FPDE. Some numerical experiments are performed in Sec-