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Asymptotic Results of Schwarz Waveform Relaxation Algorithm for Time Fractional Cable Equations

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Abstract. The equioscillation principle is an important rule to fix the parameter for the Schwarz waveform relaxation (SWR) algorithm with Robin transmission conditions. For parabolic PDEs with integer order temporal derivative, such a principle yields optimal Robin parameter, while in our previous study we found numerically that it is not always the case for time fractional PDEs: the Robin parameter determined by the equioscillation principle is sometimes far away from optimal. In this paper, by using the time fractional Cable equations as the model, we show that our previous finding does not happen occasionally but an inherent property of the SWR algorithm. Our analysis also reveals an essential difference between the asymptotic convergence rates in the overlapping and non-overlapping cases. Numerical results are provided to validate our theoretical analysis.

AMS subject classifications: 65M55, 65M12, 65M15, 65Y05

Key words: Schwarz waveform relaxation, fractional Cable equation, parameter optimization, asymptotic analysis.

1 Introduction

The Schwarz waveform relaxation (SWR) algorithm belongs to the widely used domain decomposition (DD) methods but with completely new implementation strategy for time dependent PDEs. The classical strategy employs the alternating or parallel Schwarz method to the elliptic PDEs which result from semi-discretizing the time dependent PDEs

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in time [7, 8, 35], but in the SWR framework one directly decomposes the space domain and solve all the subproblems simultaneously or alternately. The main features (or say merits) of the SWR algorithm are able to treat different subdomains numerically differently with an adapted procedure for each subdomain and the interested reader can refer to [13, 14] and [15] for the original idea of the SWR algorithm.

The configuration of the SWR algorithm is as follows. We first decompose the space domain into several subdomains and then we solve a series of time-dependent PDEs on these subdomains simultaneously or sequentially, which are formed by using the same governing equation and the same initial condition of the original PDE, but with carefully designed boundary conditions on the artificial boundaries. These boundary conditions are terminologically called *transmission conditions* (TCs), which transmit information between neighbor subdomains via iterations. The Robin TCs attract considerable attention in the past years, in which a free parameter, namely the Robin parameter p, is involved. This parameter has a significant influence on the convergence rate of the algorithm and thus finding a good Robin parameter is one of the super-priority matters.

For PDEs with integer order temporal derivative, the optimization of the Robin parameter has been investigated extensively by many authors; see, e.g., [1, 4–6, 9–12, 29, 30, 33, 39–41] and references cited therein. The main finding is the following two points. First, an ideal Robin parameter, namely $p_{\rm opt}$, is related to a min-max problem. For example, for the advection reaction diffusion equations $\partial_t u - \nu\partial_{xx}u + a\partial_x u + \mu u = f$, if we consider the SWR algorithm with overlap size $L > 0^\dagger$ such a min-max problem is

$$\min_{q>0}\max_{y\in[y_0,y_1]}\mathcal{R}_{\rm Int-order}(y,q), \quad \text{with} \quad \mathcal{R}_{\rm Int-order}(y,q) = \frac{(y-q)^2+y^2-y_0^2}{(y+q)^2+y^2-y_0^2}e^{-y}, \tag{1.1}$$

where $y_1 > y_0 > 0$ are two quantities defined by the problem and discretization parameters (see [12] for more details). With $q_{\rm opt}$ being the solution of (1.1), the optimal Robin parameter is $p_{\rm opt} = \frac{q_{\rm opt}}{2L}$. Second, at least in the asymptotic sense, the solution of the above min-max problem can be fixed by solving a nonlinear system

$$\partial_y\mathcal{R}_{\rm Int-order}(y,q) = 0, \quad \mathcal{R}_{\rm Int-order}(y_0,q) = \mathcal{R}_{\rm Int-order}(y,q). \tag{1.2}$$

The first equality in (1.2) determines the maximum of $\mathcal{R}_{\rm Int-order}(y,q)$ with respect to y and the second one implies that such a maximum equals to the value of $\mathcal{R}_{\rm Int-order}(y,q)$ at $y = y_0$. In other words, the solution of the min-max problem (1.1) satisfies an *equioscillation* principle; see Fig. 1 for illustration.

For several different types of PDEs with integer order temporal derivative, it is rigorously proved that the equioscillation equation has a unique root in the relevant interval and indeed gives the solution of the related min-max problem. For this aspect, we refer the interested reader to the work [4, 5, 12, 39] for the (advection) diffusion reaction equations, [6, 9, 10] for the Maxwell equations, [24] for the heat equations with discontinuous

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†Throughout this paper, L denotes the overlap size of the SWR algorithm. For $L > 0$ we are concerned with overlapping SWR algorithm and for $L = 0$ we are concerned with non-overlapping SWR algorithm.