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Particle Based gPC Methods for Mean-Field Models of Swarming with Uncertainty

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Abstract. In this work we focus on the construction of numerical schemes for the approximation of stochastic mean-field equations which preserve the nonnegativity of the solution. The method here developed makes use of a mean-field Monte Carlo method in the physical variables combined with a generalized Polynomial Chaos (gPC) expansion in the random space. In contrast to a direct application of stochastic-Galerkin methods, which are highly accurate but lead to the loss of positivity, the proposed schemes are capable to achieve high accuracy in the random space without loosing nonnegativity of the solution. Several applications of the schemes to mean-field models of collective behavior are reported.

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Key words: Uncertainty quantification, stochastic Galerkin, mean-field equations, swarming dynamics.

1 Introduction

An increasing number real world phenomena have been fruitfully described by kinetic and mean-field models. Particular attention has been paid in the past decade to self-organizing systems in social-economic and life sciences. Without intending to review the very huge literature on these topics, we refer the reader to [4,6,11,14,15,21,31,35,38] and the references therein.

Kinetic models may be derived in a rigorous way from microscopic particle dynamics in the limit of a large number of agents [7, 11, 23, 27, 34, 38]. It is a well known fact that the main disadvantage of the microscopic approach to capture the asymptotic behavior of interacting systems relies on the so-called curse of dimensionality. For example, if we

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consider N interacting individuals the cost is of order $O(N^2)$, becoming rapidly unaffordable in the case of large systems. For this reason, kinetic and mean-field type modeling have been developed to represent the evolution of distribution functions obtained in the asymptotic regimes, which of course become independent of the size of the system.

The introduction of uncertainty in the mathematical modeling of real world phenomena seems to be unavoidable for applications. In fact we can often have at most statistical information of the modeling parameters, which must be estimated from experiments or derived from heuristic observations [5,8,29]. Therefore, to produce effective predictions and to better understand physical phenomena, we can incorporate all the ineradicable uncertainty in the dynamics from the beginning of the modeling.

In the following a formal derivation of uncertain mean-field equations for a class of microscopic models for alignment is proposed. At the numerical level one of the most popular techniques for uncertainty quantification is based on stochastic Galerkin (SG) methods. In particular, generalized polynomial chaos (gPC) gained increasing popularity in UQ, for which spectral convergence on the random field is observed under suitable regularity assumptions [19,25,26,30,37,40,42,43]. Nevertheless, these methods need a strong modification of the original problem and when applied to hyperbolic and kinetic problems lead to the loss of some structural properties, like positivity of the solution, entropy dissipation or hyperbolicity, see [17]. Beside SG-based methods, non-intrusive approaches for UQ have been developed in recent years like the stochastic collocation (SC) methods [19, 32, 40, 41]. These methods have the nice feature to keep the structural properties of the underlying numerical solver for the deterministic problem. In this work we focus on the construction of numerical schemes which preserve the positivity of relevant statistical quantities, keeps the high accuracy typical of gPC approximations in the random space and takes advantage of the reduction of computational complexity of Monte Carlo (MC) techniques in the physical space [2, 9, 18, 34]. We investigate these Monte Carlo gPC (MCgPC) methods for mean-field type equations, which permits with a strongly reduced cost, to obtain a positive numerical approximation of expected quantities.

The rest of the paper is organized as follows. In Section 2 we introduce the microscopic models of swarming with random inputs and review some of their main properties and their mean-field limit. Section 3 is devoted to the construction of numerical methods for uncertainty quantification. We first survey some results on gPC expansions and derive the classical stochastic Galerkin scheme for the mean-field problem. Subsequently we describe the new particle based gPC approach. Finally, in Section 4 several numerical results are presented which show the efficiency and accuracy of the new Monte Carlo-gPC approach.

2 Microscopic and mean-field models with uncertainty

In the following we introduce some classical microscopic models of collective behavior [11, 16, 20] in the stochastic case characterized by random inputs. In collective motion of