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Moment Approximations and Model Cascades for Shallow Flow

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Abstract. Shallow flow models are used for a large number of applications including weather forecasting, open channel hydraulics and simulation-based natural hazard assessment. In these applications the shallowness of the process motivates depth-averaging. While the shallow flow formulation is advantageous in terms of computational efficiency, it also comes at the price of losing vertical information such as the flow's velocity profile. This gives rise to a model error, which limits the shallow flow model's predictive power and is often not explicitly quantifiable. We propose the use of vertical moments to overcome this problem. The shallow moment approximation preserves information on the vertical flow structure while still making use of the simplifying framework of depth-averaging. In this article, we derive a generic shallow flow moment system of arbitrary order starting from a set of balance laws, which has been reduced by scaling arguments. The derivation is based on a fully vertically resolved reference model with the vertical coordinate mapped onto the unit interval. We specify the shallow flow moment hierarchy for kinematic and Newtonian flow conditions and present 1D numerical results for shallow moment systems up to third order. Finally, we assess their performance with respect to both the standard shallow flow equations as well as with respect to the vertically resolved reference model. Our results show that depending on the parameter regime, e.g. friction and slip, shallow moment approximations significantly reduce the model error in shallow flow regimes and have a lot of potential to increase the predictive power of shallow flow models, while keeping them computationally cost efficient.

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1 Introduction

Mathematical shallow flow models are successfully applied in a wide range of scientific fields. Since early in the last century they constitute the basis for numerical weather

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forecasting, an historic overview is given in [20]. Another traditional application is free-surface hydraulics in rivers and channels [6, 11]. In the last decades, new applications emerged, e.g. simulation-based assessment of hazards due to gravity-driven mass movements such as snow avalanches or landslides [7, 29]. Shallow flow models are also relevant beyond geoscience applications and used to compute granular transport processes in chemical engineering, or in production engineering to model coating processes (paints, printing inks, etc), see [8, 16].

In all of these situations shallowness refers to the fact that the flow's vertical extent is much smaller than its horizontal extent. This motivates the introduction of a suitably chosen vertical average and eventually leads to a depth-averaged model of reduced complexity. Computational costs of shallow flow models are significantly lower than those of a corresponding vertically resolved free-surface flow, which is difficult to solve mainly due to the free surface. The most famous depth-averaged flow model is certainly the shallow water system. In its one dimensional formulation it is traditionally known as the Saint-Venant equations. The idea itself, however, namely deriving a shallow flow model by means of depth-averaging, is not restricted to a constitutive equation representing water and has been applied to many other fluid and granular flow rheologies [2, 15, 33]. Robust and efficient numerical methods have been developed to solve them [27, 38].

Quite naturally any shallow flow formulation comes at the price of loosing vertical information, such as information on the velocity profile. This is not critical if the velocity profile is constant throughout the flow depth and vertical acceleration can be neglected. In the geophysical flow context such a situation is often referred to as 'plug-flow'. In many realistic situations, however, velocity profiles deviate significantly from a vertically constant value. This can be observed both in large scale field experiments [21] as well as in small scale laboratory experiments [32, 34].

The computational shallow flow community encounters this obvious modeling error by introducing the concept of a shape factor. The shape factor is determined based on an assumed parametrization of the velocity profile. Simple polynomial and exponential parametrizations result in shape factors that are independent of field variables, and enter the system as additional constants [18, 22]. Though this serves as a first order correction, it is also restrictive, as the chosen parametrization, say a linear velocity profile, is assumed to be an appropriate choice throughout the duration of the flow. Data acquired during transient granular chute flow experiments again indicate that this is typically not the case and a strong regime dependency of the velocity profile can be observed [34]. The capability to model a regime dependent velocity (and shear) profile would be highly relevant in the geophysical context since it potentially allows erosion rates to be calculated, as well as the coupling forces, both of which are of special interest from an application perspective.

In our work we address this problem and propose a shallow flow formulation based on vertical moments. Moment methods proved to be a powerful tool and have been successfully applied in various scientific fields, such as in rarefied gas dynamics [14, 39]. In the shallow flow context, moment methods allow information on the vertical flow struc-