Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2017-0235

Vol. **25**, No. 3, pp. 752-780
March 2019

The Generalized Arrow-Hurwicz Method with Applications to Fluid Computation

Binbin Du and Jianguo Huang*

Key Laboratory of Scientific and Engineering Computing (Ministry of Education), School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China.

Received 26 November 2017; Accepted (in revised version) 15 January 2018

Abstract. In this paper, we first discuss the existence and uniqueness of a class of nonlinear saddle-point problems, which are frequently encountered in physical models. Then, a generalized Arrow-Hurwicz method is introduced to solve such problems. For the method, the convergence rate analysis is established under some reasonable conditions. It is also applied to solve three typical discrete methods in fluid computation, with the computational efficiency demonstrated by a series of numerical experiments.

AMS subject classifications: 65N30, 65N22, 75D05

Key words: Nonlinear saddle-point problems, the generalized Arrow-Hurwicz method, convergence rate analysis, fluid computation.

1 Introduction

This paper is intended to study existence and uniqueness of the solution to the following abstract problem and then design and analyze a generalized Arrow-Hurwicz method iteratively solving it. Afterwards, we will apply the algorithm to solve several discrete methods in fluid mechanics.

Problem P. Find $(u,p)\in V\times Q$ such that

$$\begin{cases} a_0(u,v)+N(u;u,v)-b(v,p)=\langle f,v\rangle & \forall v\in V, & (1.1)\\ b(u,q)=0 & \forall q\in Q. & (1.2)\end{cases}$$

Here, V and Q are two finite or infinite dimensional Hilbert spaces, $f\in V'$ and $\langle\cdot,\cdot\rangle$ denotes the bilinear form between the dual pair V' and V. In addition, denote by $a_0(\cdot,\cdot)$ (resp. $N(\cdot;\cdot,\cdot)$) a bounded and coercive bilinear (resp. a bounded trilinear) form over V, i.e., there exist two positive numbers α_0 and α_1 ($\alpha_0\le\alpha_1$) such that

$$\alpha_0\|v\|_V^2\le a_0(v,v)\quad\forall v\in V,\qquad a_0(u,v)\le\alpha_1\|u\|_V\|v\|_V\quad\forall u,v\in V, \tag{1.3}$$

*Corresponding author. *Email addresses:* `jghuang@sjtu.edu.cn` (J. Huang), `sjtudbb@sjtu.edu.cn` (B. Du)

and there exists a positive number $\mathcal{N}$ such that

$$|N(u;v,w)| \leq \mathcal{N}\|u\|_V\|v\|_V\|w\|_V \quad \forall u,v,w \in V. \tag{1.4}$$

We also assume $N(\cdot;\cdot,\cdot)$ is anti-symmetric with respect to the second and third components, that is, $N(\cdot;v,w) = -N(\cdot;w,v)$ for any v and w in V. Meanwhile, let $b(\cdot,\cdot)$ be a bounded bilinear form over $V \times Q$, i.e., there exists a positive constant α_2 such that

$$b(v,q) \leq \alpha_2\|v\|_V\|q\|_Q \quad \forall v \in V,\ q \in Q. \tag{1.5}$$

Throughout this paper, we use $(\cdot,\cdot)_V$ and $\|\cdot\|_V$ (resp. $(\cdot,\cdot)_Q$ and $\|\cdot\|_Q$) to stand for the inner product and the induced norm over V (resp. Q).

To the best of our knowledge, many important mathematical-physical models and their numerical methods can be described in the setting of problem P—an abstract framework. The typical examples include steady incompressible Navier-Stokes equations (cf. [18,20]), steady incompressible magnetohydrodynamics (MHD) equations (cf. [12]), and many numerical methods for solving the previous problems (cf. [15,19]).

As a matter of fact, consider the following steady incompressible MHD model:

$$\begin{cases} -R_e^{-1}\triangle \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - S_c\boldsymbol{B}\times \mathrm{curl}\boldsymbol{B} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ S_c R_m^{-1}\mathrm{curl}(\mathrm{curl}\boldsymbol{B}) - S_c\mathrm{curl}(\boldsymbol{u}\times\boldsymbol{B}) = \boldsymbol{g} & \text{in } \Omega, \\ \mathrm{div}\boldsymbol{B} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega, \\ \boldsymbol{B}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{n}\times\mathrm{curl}\boldsymbol{B} = \boldsymbol{0} & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^d$ $(d=2,3)$, $\boldsymbol{n}$ is the unit outward normal to the boundary $\partial\Omega$. Here $\boldsymbol{u}$ denotes the velocity field, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{f}$ and $\boldsymbol{g}$ the external force terms, and p the pressure field. There are three physical parameters R_e, R_m and S_c in the equations, called the hydrodynamic Reynolds number, the magnetic Reynolds number and the coupling number, respectively.

Using some notations and symbols (see Section 4.1 for details), the variational formulation of MHD model can be described as follows.

Problem P_1. Find $(\tilde{\boldsymbol{u}},p) \in \boldsymbol{W}_{0\boldsymbol{n}} \times M$ such that

$$\begin{cases} A_0(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}}) + A_1(\tilde{\boldsymbol{u}};\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}}) - d(\tilde{\boldsymbol{v}},p) = \langle \boldsymbol{F},\tilde{\boldsymbol{v}}\rangle & \forall \tilde{\boldsymbol{v}} \in \boldsymbol{W}_{0n}, \quad (1.7) \\ d(\tilde{\boldsymbol{u}},q) = 0 & \forall q \in M. \quad (1.8) \end{cases}$$

If we let $V=\boldsymbol{W}_{0n}$, $Q=M$, and for $u=\tilde{\boldsymbol{u}}$ and $v=\tilde{\boldsymbol{v}}$, write $a_0(u,v)=A_0(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}})$, $b(v,q)=d(\tilde{\boldsymbol{v}},q)$ and $N(u;u,v) = A_1(\tilde{\boldsymbol{u}};\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}})$, then problem P_1 can be viewed as a specific case of problem P. It is evident that the variational form of steady incompressible Navier-Stokes equation can be written in the form of problem P as well (cf. [9,10,18,20]). Moreover, many