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Hopf Bifurcation and Time Periodic Orbits with `pde2path` – Algorithms and Applications

Hannes Uecker*

Institut für Mathematik, Universität Oldenburg, D26111 Oldenburg, Germany.

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Abstract. We describe the algorithms used in the `Matlab` continuation and bifurcation package `pde2path` for Hopf bifurcation and continuation of branches of periodic orbits in systems of PDEs in 1, 2, and 3 spatial dimensions, including the computation of Floquet multipliers. We first test the methods on three reaction diffusion examples, namely a complex Ginzburg-Landau equation as a toy problem, a reaction diffusion system on a disk with rotational waves including stable spirals bifurcating out of the trivial solution, and a Brusselator system with interaction of Turing and Turing-Hopf bifurcations. Then we consider a system from distributed optimal control, which is ill-posed as an initial value problem and thus needs a particularly stable method for computing Floquet multipliers, for which we use a periodic Schur decomposition. The implementation details how to use `pde2path` on these problems are given in an accompanying tutorial, which also includes a number of further examples and algorithms, for instance on Hopf bifurcation with symmetries, on Hopf point continuation, and on branch switching from periodic orbits.

AMS subject classifications: 35J47, 35B22, 37M20

Key words: Hopf bifurcation, periodic orbit continuation, Floquet multipliers, partial differential equations, finite element method, reaction-diffusion, distributed optimal control.

1 Introduction

The package `pde2path` [20,72,75] has originally been developed as a continuation/bifurcation package for stationary problems of the form

$$G(u,\lambda) := -\nabla\cdot(c\otimes\nabla u)+au-b\otimes\nabla u-f=0. \tag{1.1}$$

Here $u=u(x)\in\mathbb{R}^N$, $x\in\Omega$ with $\Omega\subset\mathbb{R}^d$ some bounded domain, $d=1,2,3$, $\lambda\in\mathbb{R}^p$ is a parameter (vector), and the diffusion, advection and linear tensors c,b,a, and the nonlinearity f,

*Corresponding author. *Email address:* tt hannes.uecker@uni-oldenburg.de (H. Uecker)

$a\in\mathbb{R}^{N\times N}$ and $f\in\mathbb{R}^N$ can depend on $x,u,\nabla u$, and parameters.[1] The boundary conditions (BC) are of "generalized Neumann" form

$$\mathbf{n}\cdot(c\otimes\nabla u)+qu=g, \tag{1.2}$$

where $\mathbf{n}$ is the outer normal and again $q\in\mathbb{R}^{N\times N}$ and $g\in\mathbb{R}^N$ may depend on x, u, ∇u and parameters. These BC include zero flux BC, and a "stiff spring" approximation of Dirichlet BC via large prefactors in q and g, and periodic BC are also supported over suitable domains. Moreover, there are interfaces to couple (1.1) with additional equations, such as mass conservation, or phase conditions for considering co-moving frames, and to set up extended systems, for instance for fold point and branch point localization and continuation.

`pde2path` has been applied to various research problems, e.g., patterns in 2D reaction diffusion systems [41, 42, 64, 73, 78, 83], some problems in fluid dynamics and nonlinear optics [21, 22, 84] and in optimal control [30, 69]. Here we report on features and algorithms in `pde2path` to treat Hopf (or Poincaré-Andronov-Hopf) bifurcations and the continuation of time-periodic orbits for systems of the form

$$\partial_t u=-G(u,\lambda),\quad u=u(x,t),\ x\in\Omega\subset\mathbb{R}^d,\ d=1,2,3,\ t\in\mathbb{R}\quad (d+1\text{ dimensional problem}), \tag{1.3}$$

with G from (1.1) and BC from (1.2). Adding the time dimension makes computations more expensive, such that here we focus on 1D and 2D, and only give one 3D example to illustrate that all user interfaces are essentially dimension independent.

For general introductions to and reviews of (numerical) continuation and bifurcation we recommend [19, 27, 43, 63], and [50], which has a focus on reaction-diffusion systems. The treatment of large scale problems, typically from the spatial discretization of PDEs, including the continuation of time periodic orbits, has for instance been discussed in [47, 48, 57, 59, 68], and has recently been reviewed in [16, 52, 58]. There, the focus has been on matrix-free methods where the periodic orbits are computed by a shooting method, which can conveniently be implemented if a time-stepper for the given problem is available. In many cases, shooting methods can also be used to investigate the bifurcations from periodic orbits, and to trace bifurcation curves in parameter space, by computing the Floquet multipliers of the periodic orbits. In this direction, see in particular [45, 52, 56, 77] for impressive results in fluid problems.

[1] Originally, `pde2path` was based on the `Matlab pdetoolbox`, with $d=2$. Then, as also detailed in the `pdetoolbox` documentation, $c\in\mathbb{R}^{N\times N\times 2\times 2}$, $[\nabla\cdot(c\otimes\nabla u)]_i:=\sum_{j=1}^N[\partial_x c_{ij11}\partial_x+\partial_x c_{ij12}\partial_y+\partial_y c_{ij21}\partial_x+\partial_y c_{ij22}\partial_y]u_j$ (i^{th} component), and similarly $b\in\mathbb{R}^{N\times N\times 2}$, $a\in\mathbb{R}^{N\times N}$ with $[b\otimes\nabla u]_i:=\sum_{j=1}^N[b_{ij1}\partial_x+b_{ij2}\partial_y]u_j$, $[au]_i=\sum_{j=1}^N a_{ij}u_j$, and $f=(f_1,\cdots,f_N)$ as a column vector. For $d=1$ these formulas simplify drastically, while for $d=3$ we refer to the `OOPDE` documentation. For a given PDE, there is some freedom how to distribute terms to a,b and f; we typically set $a=0$ in the genuine PDE implementation, and only use $a\neq 0$ for assembling linearizations. Altogether we recommend the tutorials and various demo directories included with `pde2path` [72] for hints how to implement a PDE of the form (1.1) (or (1.3)) in `pde2path`.