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Superconvergence for Triangular Linear Edge Elements

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Abstract. Superconvergence for the lowest-order edge finite elements on strongly regular triangulation is studied. By the averaging technique, superconvergence of order $\mathcal{O}(h^2)$ is established at the midpoint of the interior edge for both the finite element solution and the curl of the finite element solution. Numerical results justifying our theoretical analysis are presented.

AMS subject classifications: 80M10

Key words: Maxwell's equations, superconvergence, FEM, strongly regular grid, edge elements.

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1 Introduction

Superconvergence of finite element methods (FEMs) has been an active research topic due to its strong relevance with a posteriori error estimations for the adaptive finite element method and most of the interest was devoted to elliptic and parabolic equations, see for example the surveys [2,4,7,15,25,26] and the monographs [5,9]. In regards to edge elements and their applications to Maxwell's equations, the superconvergence are limited. The first superconvergence result is due to Monk's 1994 work [23] for time-dependent Maxwell system. The integral identity technique was applied by Lin and Yan [20] to deal with the same problem once more. One order of superconvergent factor was obtained by them for k-th Nédélec elements on cubic meshes, which improved the result in [23].

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For 2-D Maxwell's equations, it was demonstrated that similar result remains true, seeing Lin [19] for the lowest order rectangular edge elements, Lin [17] for the second order rectangular edge elements and Brandts [3] for k-th triangular edge elements. Brandts' result was refined by Lin [18] in 2003: if the domain is rectangular, two order of superconvergent factor was derived for the k-th $(k\geq 1)$ Nédélec elements on the rectangular meshes. The Maxwell's equations in the aforementioned surveys were investigated in vacuum and the case of complicated medium can be also found in the literature, see for instance [16] where three popular dispersive media were considered. Note that the above mentioned results are all global superconvergence. Recently, superconvergence results at some special points are established by Huang and Li: the cubic center [14], the rectangular center [11], the midpoint of interior edge for uniform triangular mesh [13] (in these papers, the mixed FEM is used) and [10] (where the obtained superconvergence by FEM is utilized to construct an adaptive FEM method for cloaking simulation) and the midpoint of interior edge for uniform tetrahedral mesh [12]. For recent progress, we refer to the work of Chen [6], Chung [8] and Qiao [24]. A new Hybridizable Discontinuous Galerkin (HDG) method, the Staggered Discontinuous Galerkin (SDG) method and the nonuniform mixed FEM were exploited, respectively.

The superconvergence analysis on strongly regular mesh is known to be much more complicated than that on uniform mesh. It has been shown by Chen [4] that there exists superconvergence for elliptic equations on strongly regular mesh. The main goal of this paper is to transfer the superconvergence result in [4] to Maxwell's equations. Piola transformation [21], which is the covariant transformation for vector-fields, plays an essential role in our analysis.

We focus our analysis on *time harmonic Maxwell's equations* [10,12,22]:

$$\nabla\times\nabla\times\mathbf{u}-\kappa_0^2\mathbf{u}=\mathbf{f}\quad\text{in }\ \mathbf{\Omega}, \tag{1.1}$$

$$\mathbf{n}\times\mathbf{u}=0\quad\text{on }\ \partial\Omega. \tag{1.2}$$

Here, $\mathbf{u}$ represents the electric field in Ω, a Lipschitz polyhedron in $\mathbb{R}^2$. $\mathbf{f}$ stands for a given function related to the imposed current sources, which is assumed to be smooth enough. κ_0 indicates the wavenumber assumed to be real and positive and $\mathbf{n}$ denotes the outward unit normal vector field. Eq. (1.2) specifies a standard perfectly conducting boundary condition on the boundary of Ω. Let $s\geq 0$. To obtain the weak formulation of (1.1)-(1.2), we introduce the following Sobolev spaces

$$\begin{aligned}
&H(curl;\Omega)=\{\mathbf{v}\in(L^2(\Omega))^2:\nabla\times\mathbf{v}\in L^2(\Omega)\},\\
&H_0(curl;\Omega)=\{\mathbf{v}\in H(curl;\Omega):\mathbf{n}\times\mathbf{v}=0\ \text{ on }\ \partial\Omega\},\\
&H^s(curl;\Omega)=\{\mathbf{v}\in(H^s(\Omega))^2:\nabla\times\mathbf{v}\in H^s(\Omega)\}
\end{aligned}$$

equipped with the norms

$$\begin{aligned}
&||\mathbf{v}||_{H(curl;\Omega)}=(||\mathbf{v}||_0^2+||\nabla\times\mathbf{v}||_0^2)^{\frac{1}{2}},\\
&||\mathbf{v}||_{H^s(curl;\Omega)}=(||\mathbf{v}||_{H^s(\Omega)}^2+||\nabla\times\mathbf{v}||_{H^s(\Omega)}^2)^{\frac{1}{2}},
\end{aligned}$$