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A Numerical Approach for a System of Transport Equations in the Field of Radiotherapy

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Abstract. Numerical schemes for systems of transport equations are commonly constrained by a stability condition of Courant-Friedrichs-Lewy (CFL) type. We consider a system modeling the steady transport of photons and electrons in the field of radiotherapy. Naive discretizations of such a system are commonly constrained by a very restrictive CFL condition. This issue is circumvented by constructing an implicit scheme based on a relaxation approach.

We use an entropy-based moment model, namely the M_1 model. Such a system of equations possesses the non-linear flux terms of a hyperbolic system but no time derivative. The flux terms are well-defined only under a condition on the unknowns, called realizability, which corresponds to the positivity of an underlying kinetic distribution function.

The present numerical approach is applicable to non-linear systems which possess no hyperbolic operator, and it preserves the realizability property. However, the discrete equations are non-linear, and we propose a numerical method to solve such non-linear systems.

Our approach is tested on academic and practical cases in 1D, 2D, and 3D and it is shown to require significantly less computational power than reference methods.

AMS subject classifications: 35A35, 65M22, 35L65, 82C40
Key words: Implicit scheme, relaxation scheme, M_1 model, radiotherapy dose computation.

1 Introduction

The present work aims to construct a numerical solver for systems of steady transport equations emerging in the field of radiotherapy. It is a follow-up to [5, 23, 54] and it

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analyses the numerical methods used [6, 14, 50–53, 55].

The motion of energetic particles in radiotherapy can be modeled by a system of coupled linear kinetic equations over the fluences of the particles, *i.e.* over the densities, or distribution functions, of the particles in a phase space composed of position $x \in \mathbb{R}^3$, energy $\epsilon \in \mathbb{R}^+$, and direction of flight $\Omega \in \mathcal{S}^2$ on the unit sphere. Due to the high dimensionality of the phase space, solving directly such systems of equations, through either Monte Carlo methods [15, 34, 37] or discrete ordinates methods [43]; see also [44] and references therein for a review on numerical approaches for dose computation) commonly requires much higher numerical powers than the standard available in medical centers. Recent technological advances lead to the development of industrial codes based on those methods which require considerably lower computational power, *i.e.* the so-called fast Monte Carlo methods (see *e.g.* [65]) and Acuros® code [26, 48, 63].

As an alternative, in this paper, we use an angular moment extraction technique. The resulting system is under-determined, and we use an entropy minimization procedure, leading to the so-called M_1 model. We chose such a closure because it is known to preserve the main features of the underlying kinetic model (especially positivity, hyperbolicity, and entropy dissipation), and it models accurately beams of particles. This method is widely used for diverse applications in physics and biology *e.g.* in astrophysics [16,17,30], radiative transfer [21, 56], in fluid dynamics [29, 42, 45], for semiconductors [31, 57] or chemotaxis [7] modeling, and showed a considerable reduction of the numerical costs.

Numerical approaches for solving moment equations are typically constrained by a stability condition. Such a condition becomes very restrictive when considering low density media. Typically, the step size (see [5, 54] or Section 3 below) needs to be taken proportional to the minimum density in the medium and therefore many steps are required. This problem was first studied for application in radiotherapy in [5] and it was circumvented by the use of a clever change of variables. The previous work [54] showed another approach based on a relaxation method (based on [1,11,47], see also recent work [18]) and on the method of characteristics. However, both those approaches are inappropriate to model the motion of photons. Indeed, those numerical schemes are applicable only to hyperbolic systems, but the transport of photons is ill-modeled by such equations [49].

In the present paper, we present an implicit scheme based on a relaxation method, preserving the realizability property and efficient with large steps. However, the discretized equations are non-linear, and we construct an iterative solver to solve such equations.

The paper is constructed as follows. In the next section, models of transport of photons and electrons are presented, first a kinetic model, then the angular moment extraction is described. A first numerical scheme is described for 1D problems in Section 3, an iterative algorithm adapted to this scheme is constructed and tested on an academic test case. This numerical scheme is completed and adapted to multi-D problems in Section 4 and tested on academic test cases in 2D and 3D. Section 5 is devoted to conclusion.