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Reconstructed Discontinuous Galerkin Methods for Hyperbolic Diffusion Equations on Unstructured Grids

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Abstract. Reconstructed Discontinuous Galerkin (rDG) methods are presented for solving diffusion equations based on a first-order hyperbolic system (FOHS) formulation. The idea is to combine the advantages of the FOHS formulation and the rDG methods in an effort to develop a more reliable, accurate, efficient, and robust method for solving the diffusion equations. The developed hyperbolic rDG methods can be made to have higher-order accuracy than conventional DG methods with fewer degrees of freedom. A number of test cases for different diffusion equations are presented to assess accuracy and performance of the newly developed hyperbolic rDG methods in comparison with the standard BR2 DG method. Numerical experiments demonstrate that the hyperbolic rDG methods are able to achieve the designed optimal order of accuracy for both solutions and their derivatives on regular, irregular, and heterogeneous girds, and outperform the BR2 method in terms of the magnitude of the error, the order of accuracy, the size of time steps, and the CPU times required to achieve steady state solutions, indicating that the developed hyperbolic rDG methods provide an attractive and probably an even superior alternative for solving the diffusion equations.

AMS subject classifications: 76M10, 35L55

Key words: Discontinuous Galerkin methods, first-order hyperbolic system, unstructured grids.

1 Introduction

The discontinuous Galerkin (DG) methods [2, 4, 5, 10–12, 17, 19, 20, 26–30, 48, 49, 52] have recently become popular for the solution of systems of conservation laws, owing to their attractive features, such as flexibility to handle complex geometry, compact stencil for higher-order reconstruction, and amenability to parallelization and *hp*-adaptation.

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Nowadays, they are widely used in computational fluid dynamics, computational acoustics, and computational magneto-hydrodynamics. The DG methods combine two advantageous features commonly associated to the finite element (FE) and finite volume (FV) methods. As in classical finite element methods, the order of accuracy is obtained by means of high-order polynomial approximation within an element rather than by wide stencils as in the finite volume methods. The physics of wave propagation is, however, accounted for by solving the Riemann problems that arise from the discontinuous representation of the solution at element interfaces. In this respect, the DG methods (DGMs) are therefore similar to the finite volume methods. However, the DGMs have a number of their own weaknesses. In particular, how to effectively control spurious oscillations in the presence of strong discontinuities, how to reduce the computing costs for the DGMs, and how to efficiently solve elliptic problems or discretize diffusion terms in the parabolic equations are three interesting and challenging research topics in the DGMs.

The DGMs have been recognized as expensive in terms of both computational costs and storage requirements. Indeed, compared to the FE and FV methods, the DGMs require solutions of systems of equations with more unknowns for the same grids. In order to reduce high costs associated with the DGMs, Dumbser et al. [14–16] have introduced a new family of reconstructed DGM, termed PnPm schemes and referred to as rDG(PnPm) in this paper, where Pn indicates that a piecewise polynomial of degree of n is used to represent a DG solution, and Pm represents a reconstructed polynomial solution of degree of m ($m \geq n$) that is used to compute the fluxes. The rDG(PnPm) schemes [31,32,35,53,54] are designed to enhance the accuracy of the DGM by increasing the order of the underlying polynomial solution. The beauty of rDG(PnPm) schemes is that they provide a unified formulation for both FVM and DGM, and contain both classical FVM and standard DGM as two special cases of rDG(PnPm) schemes. When $n=0$, i.e. a piecewise constant polynomial is used to represent a numerical solution, rDG(P0Pm) is nothing but classical high order FV schemes, where a polynomial solution of degree m ($m \geq 1$) is reconstructed from a piecewise constant solution. When $m=n$, the reconstruction reduces to the identity operator, and rDG(PnPn) scheme yields a standard DG(Pn) method. For $n>0$, and $m>n$, a new family of numerical methods from third-order of accuracy upwards is obtained. A Hierarchical WENO-based rDG method [34, 36] is designed not only to reduce the high computing costs of the DGM, but also to avoid spurious oscillations in the vicinity of strong discontinuities, thus effectively overcoming the first two shortcomings of the DG methods.

The DGMs are indeed a natural choice for the solution of the hyperbolic equations, such as the compressible Euler equations. However, the DG formulation is far less certain and advantageous for elliptic problems or parabolic equations such as the compressible Navier-Stokes equations, where diffusive fluxes exist and which require the evaluation of the solution derivatives at the interfaces. Taking a simple arithmetic mean of the solution derivatives from the left and right is inconsistent, because it does not take into account a possible jump of the solutions. A number of numerical methods have been proposed in the literature to address this issue, such as those by Bassi and Rebay [3, 5, 6],