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A Highly Efficient Numerical Method for Rotating Oceanographic Flows Modeled by Saint-Venant System with Coriolis Forces

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Abstract. In order to overcome the strong limitation of the computational cost for classic explicit schemes in quasi-geostrophic limit typically involved in oceanographic flows, in this study, a highly efficient numerical method for rotating oceanographic flows modeled by Saint-Venant system with Coriolis forces is developed. To efficiently deal with the rotating flows in the low Rossby and Froude number regime, the core idea is splitting fast varying flux terms into stiff and non-stiff parts, and implicitly approximating the fast dynamic waves using central difference method with an iteration algorithm and explicitly approximating slow dynamic waves using a finite-volume hyperbolic solver with minmod limiter. The proposed approach has a second order convergence rate in the quasi-geostrophic limit. The temporal evolution is approximated using a high-order implicit-explicit Runge-Kutta method. The proposed semi-implicit scheme is proved to be uniformly asymptotically consistent in the quasi-geostrophic limit when Froude and Rossby numbers→0. The proposed numerical methods are finally verified by numerical experiments of rotating shallow flows. The tests show that the proposed numerical scheme is stable and accurate with any grid size with low Rossby and Froude numbers, which leads to significant reduction of the computational cost comparing with classic explicit schemes.

AMS subject classifications: 35Q35, 76U05, 76N99, 76M12, 65M06

Key words: Saint-Venant system, implicit-explicit, Coriolis, low Rossby number, low Froude number, finite-volume method, quasi-geostrophic limit.

1 Introduction

Oceanographic flows generally take place over horizontal length scales which are much larger than vertical length scale, thus, they can be demonstrated in oceanography using

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the Saint-Venant system which describes a thin layer free surface flow of constant density under hydrostatic assumption over a rigid bottom. Two dimensional (2D) Saint-Venant system with Coriolis forces over flat bottom reads

$$h_t+\nabla\cdot(h\boldsymbol{u})=0, \tag{1.1}$$

$$(h\boldsymbol{u})_t+\nabla\cdot(h\boldsymbol{u}\otimes\boldsymbol{u})+\nabla\left(\frac{g}{2}h^2\right)=fh\boldsymbol{u}^\perp, \tag{1.2}$$

where t is time, h is the water depth, $\boldsymbol{u}=(u,v)$ is the velocities and $\boldsymbol{u}^\perp=(v,-u)$ is the orthogonal velocity in which u and v are horizontal velocity components in the x- and y-directions, respectively, g is the gravitational acceleration, and f is the Coriolis parameter.

Applying the standard non-dimensionalization, one first introduce four physical parameters which are a reference length scale l_{ref}, a reference Coriolis parameter f_{ref}, a reference depth selected to be $h_{ref}=l_{ref}^2 f_{ref}^2/g$, a reference flow velocity u_{ref} and a reference time scale $t_{ref}=l_{ref}/u_{ref}$. Considering one height scale h_{ref} in the current study, according to the Buckingham π-theorem, three dimensionless group parameters, i.e. the Froude number(Fr), the Strouhal number(Sr) and Rossby number(Ro), are introduced by defining that

$$\mathrm{Fr}=\frac{u_{ref}}{\sqrt{gh_{ref}}}=\varepsilon,\quad \mathrm{Sr}=\frac{l_{ref}}{t_{ref}u_{ref}}=1,\quad \mathrm{Ro}=\frac{u_{ref}}{l_{ref}f_{ref}}=\varepsilon. \tag{1.3}$$

Correlations among the dimensionless group parameters in (1.3) define particular distinguished limits. In order to simulate geostrophic flows which may be modeled by a rotating shallow water wave with a frequency of zero, it is reasonable to choose $\mathrm{Sr}=1$ for a long-term dynamics of the system. Moreover, the current study focus on the quasi-geostrophic distinguished limit in which the Froude and Rossby numbers are very small, the pressure gradient and the Coriolis force are almost balanced, and the variation of the surface perturbation is very mild compared to the water depth. Therefore, throughout, the present study considers the cases in which the inertial force are systematically small compared with the Coriolis force and the speed of the gravity waves, and the reference asymptotic expansion parameter $\varepsilon=\mathrm{Fr}\ll 1$ is introduced in (1.3) to define the low Rossby number and low Froude number in the quasi-geostrophic limits.

Substituting dimensionless variables $\widehat{x}:=x/l_{ref}$, $\widehat{y}:=y/l_{ref}$, $\widehat{h}:=h/h_{ref}$, $\widehat{u}:=u/u_{ref}$, $\widehat{v}:=v/u_{ref}$, $\widehat{t}:=t/t_{ref}$ and $\widehat{f}:=f/f_{ref}$ into (1.1)-(1.2), and then dropping the hat notions, one can obtain the non-dimensional Saint-Venant system:

$$h_t+\nabla\cdot(h\boldsymbol{u})=0, \tag{1.4}$$

$$(h\boldsymbol{u})_t+\nabla\cdot(h\boldsymbol{u}\otimes\boldsymbol{u})+\nabla\left(\frac{h^2}{2\varepsilon^2}\right)=\frac{1}{\varepsilon}h\boldsymbol{u}^\perp. \tag{1.5}$$

Notice that, it is assumed in this study that $f_{ref}=f$ so that $\widehat{f}=1$.

Classic explicit numerical schemes for the Saint-Venant Eqs. (1.1)-(1.2) have been intensely studied, see e.g. [4, 5, 7, 12–17, 21–23, 25–29, 34, 37, 40]. However, these explicit