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Capturing Near-Equilibrium Solutions: A Comparison between High-Order Discontinuous Galerkin Methods and Well-Balanced Schemes

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Abstract. Equilibrium or stationary solutions usually proceed through the exact balance between hyperbolic transport terms and source terms. Such equilibrium solutions are affected by truncation errors that prevent any classical numerical scheme from capturing the evolution of small amplitude waves of physical significance. In order to overcome this problem, we compare two commonly adopted strategies: going to very high order and reduce drastically the truncation errors on the equilibrium solution, or design a specific scheme that preserves by construction the equilibrium exactly, the so-called well-balanced approach. We present a modern numerical implementation of these two strategies and compare them in details, using hydrostatic but also dynamical equilibrium solutions of several simple test cases. Finally, we apply our methodology to the simulation of a protoplanetary disc in centrifugal equilibrium around its star and model its interaction with an embedded planet, illustrating in a realistic application the strength of both methods.

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1 Introduction

Hyperbolic balance laws are used to describe many dynamical problems in natural sciences. They are defined as a set of conservation laws with associated source terms, which

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model the production or destruction of the corresponding conserved quantity. Many physical systems of scientific interest can be described by a system of hyperbolic conservation laws with source terms, or in short, hyperbolic balance laws.

Hyperbolic balance laws are particularly challenging because they feature equilibrium solutions that result from the exact cancellation of the divergence of the flux and the source terms in these equations. Small truncation errors can perturb this equilibrium solution, leading to the production of spurious waves that can dominate over the real waves that control the physics of the problem at hand.

For example, for the case of the inviscid Euler equations with a gravity source term (also known as the Euler-Poisson system), hydrostatic steady states are important in, for example, hydraulics [3, 5, 6] and astrophysics [20, 21, 29]. The difficulty here is to capture properly sound waves, gravity waves or convective flows, whose amplitude can be comparable to the truncation errors of a second order method and a reasonable grid resolution.

General steady states with non constant velocity fields are also found to be important in planetary sciences, namely in the early stages of protoplanetary discs, where the source term models the gravity of a central star [28], and is balanced by the centrifugal and pressure forces. The challenge here is to be able to resolve the interaction of a small planet with the gaseous disc, leading to the formation of a small amplitude spiral wave that can be dominated by the truncation errors of the equilibrium solution. In this context, the classical approach is to use a cylindrical mesh, reducing drastically discretisation errors along circular orbits. It is however desirable to find a solution on a Cartesian mesh, as it allows to deal with more general cases which are not strictly axisymmetric.

In summary, solving for such flows which are close to equilibrium can be very challenging for a naive, low order numerical method on a mesh not necessarily adapted to the geometry of the equilibrium solution as the truncation error incurred while solving the steady state can be larger than the small amplitude waves of interest.

There are nowadays many practical numerical methods with very low truncation errors. A class of such methods are the so-called discontinuous Galerkin (DG) methods [1]. These methods, at least for smooth and regular problems, can be made as accurate as desired. This means that, at least in principle, the amplitude of the truncation errors can be reduced to an arbitrarily small value. This requires an appropriate way to implement the source terms in the DG formalism [7, 18]. This also requires the use of a high enough resolution mesh to capture the equilibrium solution, which translates into higher computational cost for higher order solutions.

There is another strategy that allows one to use a low-order method, while capturing almost exactly the equilibrium solution. This is called the well-balanced approach (introduced in detail [17]), which is concerned with numerical schemes that satisfy the discrete equivalent of an underlying steady state, effectively, taking into account the existence of a steady state (or near steady state) solution.

The natural question is thus whether exact well-balancedness is required in practice or if methods that solve the PDE (including the source term) need only to be very accu-