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A Novel Method for Solving Time-Dependent 2D Advection-Diffusion-Reaction Equations to Model Transfer in Nonlinear Anisotropic Media

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Abstract. This paper presents a new numerical technique for solving initial and boundary value problems with unsteady strongly nonlinear advection diffusion reaction (ADR) equations. The method is based on the use of the radial basis functions (RBF) for the approximation space of the solution. The Crank-Nicolson scheme is used for approximation in time. This results in a sequence of stationary nonlinear ADR equations. The equations are solved sequentially at each time step using the proposed semi-analytical technique based on the RBFs. The approximate solution is sought in the form of the analytical expansion over basis functions and contains free parameters. The basis functions are constructed in such a way that the expansion satisfies the boundary conditions of the problem for any choice of the free parameters. The free parameters are determined by substitution of the expansion in the equation and collocation in the solution domain. In the case of a nonlinear equation, we use the well-known procedure of quasilinearization. This transforms the original equation into a sequence of the linear ones on each time layer. The numerical examples confirm the high accuracy and robustness of the proposed numerical scheme.

AMS subject classifications: 65N35, 65N40, 65Y20

Key words: Advection diffusion reaction, time-dependent, fully nonlinear, anisotropic media, Crank-Nicolson scheme, meshless method.

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1 Introduction

The governing equation of a variety of physical problems in engineering and science is expressed by the advection-diffusion-reaction (ADR) equation. The ADR equation is a second order parabolic partial differential equation (PDE). In this paper we consider the ADR equation in the form:

$$\frac{\partial C(\mathbf{x},t)}{\partial t}=div[\boldsymbol{Q}(\mathbf{x},t,C)]+\nabla\cdot(\mathbf{a}(\mathbf{x},t)C)+R\left(\mathbf{x},C,C_x,C_y,t\right)=0,\quad \mathbf{x}=(x_1,x_2)\in\Omega, \tag{1.1}$$

where $C(\mathbf{x},t)$ is the variable of interest (such as the concentration of pollutant for mass transfer and the temperature for heat transfer etc). The diffusion term $div[\boldsymbol{Q}(\mathbf{x},t,C)]$ describes the micro transport of $C(\mathbf{x},t)$ due to its gradients. Here $\boldsymbol{Q}(\mathbf{x},t,C)$ is the flux vector of $C(\mathbf{x},t)$

$$\boldsymbol{Q}(\mathbf{x},t,C)=\widehat{D}(\mathbf{x},C,t)\nabla C(\mathbf{x},t). \tag{1.2}$$

In the general case of anisotropic media, the diffusivity $\widehat{D}$ is the second order tensor which can be represented as a symmetric matrix whose entries are bounded functions:

$$\widehat{D}(\mathbf{x},C,t)=\begin{pmatrix} D_{11}(\mathbf{x},C,t),D_{12}(\mathbf{x},C,t)\\ D_{21}(\mathbf{x},C,t),D_{22}(\mathbf{x},C,t)\end{pmatrix}, \tag{1.3}$$

where $D_{21}=D_{12}$, $D_{11}D_{22}>D_{12}D_{21}$ from Onsagar's reciprocity relation which provides the elliptic type of the differential operator in the right hand side of the equation. The advection term $\nabla\cdot(\mathbf{a}(\mathbf{x},t)C)$ describes the macro transfer of the quantities, where $\mathbf{a}(\mathbf{x},t)=(a_1(\mathbf{x},t),a_2(\mathbf{x},t))$ is the velocity of the media, i.e., is the velocity field that the quantity C is moving with. For incompressible media, the velocity vector satisfies the condition $div[\mathbf{a}(\mathbf{x})]=0$. The term $R\left(\mathbf{x},C,C_x,C_y,t\right)$ describes "sources" or "sinks" of $C(\mathbf{x},t)$ (results of the chemical reactions, heat sources etc.). Below we represent this term in the form

$$R\left(\mathbf{x},C,C_x,C_y,t\right)=q\left(\mathbf{x},C,C_x,C_y,t\right)-f(\mathbf{x},t).$$

In engineering applications, the ADR equation expresses heat transfer and transport of mass and chemicals into porous or nonporous media [1]. The systems of ADR equations are common mathematical models used to describe the transport of contamination in atmosphere [2] and groundwater [3], radiation of microwaves [4], climate modelling [5], batch culture of biofilm [6] and wetland hydrology [7]. In most cases it is difficult and also time consuming to solve such problems explicitly. Therefore, it is necessary to obtain their approximate solutions by using some efficient numerical methods. The finite difference method (FDM) and the finite element (FEM) techniques [8] are classical tools for the numerical modeling of the ADR problem. A detailed review of the classic methods involving FEMs and FDMs can be found in [9]. Recent developments of these techniques can be found in [10, 11] and references therein. Spectral methods