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Asymptotic Preserving Spectral Deferred Correction Methods for Hyperbolic Systems with Relaxation

Chong Sun[1] and Yinhua Xia[1,*]

[1] *School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China.*

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Abstract. In this paper, we consider the semi-implicit spectral deferred correction (SDC) methods for hyperbolic systems of conservation laws with stiff relaxation terms. The relaxation term is treated implicitly, and the convection terms are treated by explicit schemes. The SDC schemes proposed are asymptotic preserving (AP) in the zero relaxation limit and can be constructed easily and systematically for any order of accuracy. Weighted essentially non-oscillatory (WENO) schemes are adopted in spatial discretization to achieve high order accuracy. After a description of the asymptotic preserving property of the SDC schemes, several applications will be presented to demonstrate the stiff accuracy and capability of the schemes.

AMS subject classifications: 65M70, 65B05, 35L45

Key words: Spectral deferred correction methods, asymptotic preserving schemes, hyperbolic systems with relaxation, stiff systems, weighted essentially non-oscillatory schemes.

1 Introduction

In recent years, the research on the hyperbolic system with relaxation has become an active field, due to its importance on physical problems [1,10,25]. For example, hyperbolic systems with relaxation appear in shallow water, traffic flows, hydrodynamical models for semiconductors and so on.

Hyperbolic systems with relaxation are described by stiff systems of differential equations in the form

$$\partial_t U + \nabla \cdot F(U) = \frac{1}{\epsilon} R(U), \tag{1.1}$$

where $U \in \mathbb{R}^N, F, R: \mathbb{R}^N \to \mathbb{R}^N$ and $\epsilon > 0$ is the stiffness or relaxation parameter. Especially, in one space dimension the system has the form

$$\partial_t U + \partial_x F(U) = \frac{1}{\epsilon} R(U). \tag{1.2}$$

*Corresponding author. *Email addresses:* `sc1500@mail.ustc.edu.cn` (C. Sun), `yhxia@ustc.edu.cn` (Y. Xia)

The system is hyperbolic if the Jacobian matrix $\partial_U F(U)$ has only real eigenvalues and is diagonalizable for every U.

The development of efficient numerical schemes for such systems is challenging, since hyperbolic equations with small scales lead to various different asymptotic regimes, where classical numerical approximations become prohibitively expensive. Fortunately, asymptotic preserving (AP) schemes are efficient in these asymptotic regimes. This approach has its origin in capturing steady-state solution for neuron transport in the diffusive regime [22, 23]. Since the 90s of last century, the AP schemes have been developed for kinetic and hyperbolic equations and research on robust AP schemes has made great progress [8, 13, 18–20]. The designing principle [20] of AP schemes is to preserve, at the discrete level, the asymptotic limit that drives one (the microscopic) equation to its asymptotic (macroscopic) equation. An AP scheme solves the microscopic equation, instead of using a multiphysics approach that couples different physical laws at different scales, making the computational methods efficient.

Recently developed splitting methods [20] and the implicit-explicit Runge-Kutta (IMEX-RK) methods [28] belong to the AP schemes, which have been widely used for such problems. Splitting methods are attractive because of their simplicity and robustness. Strang splitting schemes provide second order accuracy if each step is at least second order accurate [32]. For stiff problem, this property is maintained under fairly mild assumptions [17]. However, Strang splitting applied to hyperbolic systems with relaxation reduces to first order accuracy when the problem becomes stiff. The reason is that the kernel of the relaxation operator is nontrivial, which corresponds to a singular matrix in the linear case, and therefore the assumptions in [17] are not satisfied. It is also difficult to obtain higher order accuracy even in non-stiff regimes. Fortunately, the implicit-explicit Runge-Kutta schemes [8, 18, 28, 38] overcome this difficulty, providing basically the same advantages of the splitting schemes, without the drawback of the order restriction. In [28] up to the third order accurate IMEX-RK methods has been developed, which are strong-stability-preserving (SSP) for the limiting system of conservation laws. The SSP Runge-Kutta methods were originally referred to as the total variation diminishing (TVD) Runge-Kutta methods, see [14, 15, 31]. At the same time, although the implicit-explicit Runge-Kutta schemes can be constructed for any high order accuracy, the correlation coefficient of high order accuracy is not very easy to obtain. Recently there is a one-step space-time integration method [7] has been developed for the non-conservative hyperbolic systems with stiff terms.

Furthermore, in this paper, we will present the AP schemes based on semi-implicit spectral deferred correction (SDC) methods for hyperbolic systems of conservation laws with stiff relaxation terms. The SDC methods are constructed by Dutt, Greengard and Rokhlin [12]. They are the variants of the deferred correction (DC) methods [2], which apply the DC approach to the integral formulation of the error equation and adopt the spectral collocation points in the quadrature rule. When the quadrature nodes are uniform, the SDC method is called the integral deferred correction (InDC) method [11]. There are various DC methods with different implementation strategies and applications, e.g. the