Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2018-0157

Vol. **26**, No. 2, pp. 579-598
August 2019

A Numerical Study of the 3-Periodic Wave Solutions to Toda-Type Equations

Yingnan Zhang[1], Xingbiao Hu[2,3], Yi He[4] and Jianqing Sun[5,*]

[1] *School of Mathematical Sciences, Nanjing Normal University, Nanjing, P.R. China.*
[2] *LSEC, Institute of Computational Mathematics and Scientific Engineering Computing, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing, P.R. China.*
[3] *School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, P.R. China.*
[4] *Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan, P.R. China.*
[5] *School of Mathematical Sciences, Ocean University of China, Qingdao, P.R. China.*

Received 10 June 2018; Accepted (in revised version) 3 September 2018

Abstract. In this paper, we present an efficient numerical scheme to calculate N-periodic wave solutions to the Toda-type equations. The starting point is the algebraic condition for having N-periodic wave solutions proposed by Akira Nakamura. The basic idea is to formulate the condition as a nonlinear least square problem and then use the Gauss-Newton method to solve it. By use of this numerical scheme, we calculate the 3-periodic wave solutions to some discrete integrable equations such as the Toda lattice equation, the Lotka-Volterra equation, the differential-difference KP equation and so on.

AMS subject classifications: 37K10, 39A10, 39A23, 65H10, 65Q10

Key words: Toda-type equation, N-periodic wave solution, Riemann's θ-function, Gauss-Newton method.

1 Introduction

In this paper, we focus on numerical calculation of N-periodic wave solutions to some completely integrable differential-difference equations or difference-difference equations. The periodic solution mentioned here represents the solutions expressed in terms of Riemann's θ-functions [1]. These kinds of solutions are periodic analogues of soliton solutions, and in general case the N-periodic wave solution is a periodic generalization of the

*Corresponding author. *Email addresses:* `ynzhang@njnu.edu.cn` (Y. Zhang), `hxb@lsec.cc.ac.cn` (X. Hu), `heyi@lsec.cc.ac.cn` (Y. He), `sunjq@lsec.cc.ac.cn` (J. Sun)

N-soliton solution or multiple collision of N solitons [2]. In the KdV equation, this kind of solutions are also called finite-genus solutions or finite-gap solutions [3,4].

Much work has already been done on periodic waves in continuous case. The pioneering work was made by Novikov and Dubrovin [3–5], Lax [6], Its and Matveev [7], McKean and van Moerbeke [8] in 1970s. After that, some classical methods such as the inverse scattering method [9–11], the algebro-geometric approach [1,12–17] and the direct method [2,18–22], are applied to solve periodic waves. However, comparing with solitary waves, the periodic waves are more complicated and it is difficult to give some detailed explicit expressions. Therefore, many researchers turn to numerical calculations. Recent work includes the numerical approach via Riemann-Hilbert problem [23,24] and spectral method [25–27]. Here we will study the periodic waves numerically based on the direct method [2,18,19,28].

In [2,18], Nakamura first proposed a condition for having N-periodic waves to nonlinear evolution equation which can be reduced to certain types of bilinear equations, such as KdV, mKdV, NLS and some other equations. After that, Hirota [19] suggested researchers investigate whether the soliton equations written in the KdV-type bilinear form exhibit 3-periodic wave solutions or not by this condition. Recently, in [29], the authors further applied the condition together with Gauss-Newton method to the KdV-type equations and gave some positive answers to the existence of 3-periodic waves to several KdV-type equations: the KdV equation, the Sawada-Kotera equation, the Boussinesq equation, the Ito equation [30], the Hietarinta equation [31] and the $(2+1)$-dimensional KP equation.

In this paper, we will apply the condition to the Toda-type equations and give a numerical procedure to calculate their 3-periodic wave solutions. Here "*Toda-type*"refers to equations possessing a similar bilinear form to that of the Toda lattice equation. The famous Toda lattice equation was first proposed as a lattice model with exponential interactions by Toda [32–34]. As a completely integrable equation, it has been studied a great deal. Toda showed that it admits rigorous periodic solutions and soliton solutions [35]. Date studied its quasi-periodic wave solutions by using the inverse scattering method [11]. Recently in [36], Kodama and his collaborators studied its quasi-periodic wave solutions via the hyperelliptic sigma functions and in [37], Geng and his collaborators studied a four-component Toda lattice equation and its quasi-periodic solutions. It is known that the Toda lattice equation has a form [28]

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log(1+V_n)=V_{n+1}+V_{n-1}-2V_n, \tag{1.1}$$

which can be transformed into the bilinear equation [19]

$$\left[D_t^2-4\lambda\sinh^2(D_n/2)+2(1-\lambda)\right]f_n\cdot f_n=0, \tag{1.2}$$

through the transformation

$$V_n=\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log f_n, \tag{1.3}$$