Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2018-0152

Vol. **26**, No. 2, pp. 599-630
August 2019

Second Order Finite Volume Scheme for Euler Equations with Gravity which is Well-Balanced for General Equations of State and Grid Systems

Jonas P. Berberich[1], Praveen Chandrashekar[2], Christian Klingenberg[1,*] and Friedrich K. Röpke[3,4]

[1] *Department of Mathematics, Univeristy of Würzburg, 97074 Würzburg, Germany.*
[2] *TIFR Center for Applicable Mathematics, Bangalore 560065, India.*
[3] *Zentrum für Astronomie der Universität Heidelberg, Institut für Theoretische Astrophysik, Philosophenweg 12, 69120 Heidelberg, Germany.*
[4] *Heidelberger Institut für Theoretische Studien, Schloss-Wolfsbrunnenweg 35, 69118 Heidelberg, Germany.*

Communicated by Kun Xu

Received 8 June 2018; Accepted (in revised version) 2 September 2018

Abstract. We develop a second order well-balanced finite volume scheme for compressible Euler equations with a gravitational source term. The well-balanced property holds for arbitrary hydrostatic solutions of the corresponding Euler equations without any restriction on the equation of state. The hydrostatic solution must be known a priori either as an analytical formula or as a discrete solution at the grid points. The scheme can be applied on curvilinear meshes and in combination with any consistent numerical flux function and time stepping routines. These properties are demonstrated on a range of numerical tests.

AMS subject classifications: 76M12, 65M08, 65M20, 35L65, 76N15, 76E20, 85-08

Key words: Finite volume methods, well-balancing, compressible Euler equations with gravity.

1 Introduction

The Euler equations with gravitational source term are used to model the flow of gases in different fields of physical sciences. Examples include weather prediction, climate modeling, and several astrophysical application such as the modeling of stellar interiors.

*Corresponding author. *Email addresses:* `klingen@mathematik.uni-wuerzburg.de` (C. Klingenberg), `jonas.berberich@mathematik.uni-wuerzburg.de` (J. P. Berberich), `praveen@tifrbng.res.in` (P. Chandrashekar), `friedrich.roepke@h-its.org` (F. K. Röpke)

In many of these applications, the gas state is close to a hydrostatic solution. Hydrostatic solutions are non-trivial steady state solutions of the Euler equations with gravity which can be described using the differential equation $\nabla p(\mathbf{x}) = -\rho(\mathbf{x})\nabla(\phi(\mathbf{x}))$, where p is the gas pressure, ρ is the gas density and ϕ is a given external gravitational potential. In order to resolve the flow dynamics, which can be seen as small perturbations of the hydrostatic solution, one has to be able to maintain the corresponding hydrostatic solution with a sufficiently small error. Conventional methods introduce a significant discretization error when trying to compute small perturbations of the hydrostatic solution, especially on coarse meshes. Since the computational effort using sufficiently fine meshes can be too high, especially in three-dimensional simulations, special numerical techniques for this problem have been developed called well-balanced schemes. Well-balanced schemes are able to maintain hydrostatic solutions close to machine precision even on coarse meshes.

Well-balanced schemes have been developed for the well-known shallow water equations with non-flat bottom topography. The equation describing steady state solutions in the shallow water equations is given in an explicit algebraic form which favors the development of well-balanced schemes. Some examples are [1, 31]. There are also well-balanced schemes for related models, like e.g. the Ripa model [13, 37]. More recently, well-balanced schemes for Euler equations with gravitational source term have been developed. This is more delicate than for shallow water equations since the hydrostatic solutions are given implicitly via a differential equation. For different equations of state (EoS) different hydrostatic solutions can be found. This led to the development of well-balanced schemes which are restricted to certain EoS and classes of hydrostatic solutions. Early work on this topic has been conducted by Cargo and Le Roux [5]. LeVeque and Bale [23] applied a quasi-steady wave-propagation algorithm on the Euler equations with gravitational source term to maintain isothermal hydrostatic solutions numerically. This method has been expanded to isentropic solutions in [24]. Even for high order schemes well-balancing is necessary if solutions close to a hydrostatic solution are computed [40]. A high order well-balanced scheme for isothermal hydrostatic solutions is introduced in [40]. The scheme includes a modified weighted essentially non-oscillatory (WENO) reconstruction and a suitable way to discretize the source term. Based on this idea, a non-staggered central scheme for the same class of hydrostatic solutions has been proposed in [38]. Compact reconstruction WENO methods are applied to achieve well-balancing in [17]. Discontinuous Galerkin (DG) well-balanced methods have been developed in [8, 25, 26]. The well-balanced method proposed in [9] is based on a reformulation of the Euler equations with gravity discretized using a central scheme.

Another approach to achieve well-balancing for Euler with gravitational source term is the development of well-balanced relaxation schemes, see [11, 12, 35] and references therein. Here stable approximate Riemann solvers are constructed for well-balancing. Methods based on hydrostatic reconstruction were first developed by Audusse et al. [1] for the shallow water equations. Later they have been adapted for the Euler equations with gravitational source term, see e.g. [7]. Early applications for weather prediction can be seen in [3].