Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2018-0191

Vol. **26**, No. 3, pp. 749-767
September 2019

A Nonlinear Elimination Preconditioner for Fully Coupled Space-Time Solution Algorithm with Applications to High-Rayleigh Number Thermal Convective Flow Problems

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Communicated by Kun Xu

Received 26 July 2018; Accepted (in revised version) 19 October 2018

Abstract. As the computing power of the latest parallel computer systems increases dramatically, the fully coupled space-time solution algorithms for the time-dependent system of PDEs obtain their popularity recently, especially for the case of using a large number of computing cores. In this space-time algorithm, we solve the resulting large, space, nonlinear systems in an all-at-once manner and a robust and efficient nonlinear solver plays an essential role as a key kernel of the whole solution algorithm. In the paper, we introduce and study some parallel nonlinear space-time preconditioned Newton algorithm for the space-time formulation of the thermal convective flows at high Rayleigh numbers. In particular, we apply an adaptive nonlinear space-time elimination preconditioning technique to enhance the robustness of the inexact Newton method, in the sense that an inexact Newton method can converge in a broad range of physical parameters in the multi-physical heat fluid model. In addition, at each Newton iteration, we find an appropriate search direction by using a space-time overlapping Schwarz domain decomposition algorithm for solving the Jacobian system efficiently. Some numerical results show that the proposed method is more robust and efficient than the commonly-used Newton-Krylov-Schwarz method.

AMS subject classifications: 76D05, 65F08, 49M15, 65M55, 68W10

Key words: Fluid flow, heat transfer, nonlinear elimination, space-time, domain decomposition, parallel computing.

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1 Introduction

Time-marching methods and their variants are popular numerical algorithms for solving the system of nonlinear time-dependent partial differential equations (PDEs) [32,34]. After spatial discretization, the basic idea of the time-marching method is to integrate numerically step by step along the temporal direction from an initial state. As a result, the recent development of parallel solution algorithms based on the time-marching methods focuses mainly on the parallelization within each time step, and that is purely sequential between time steps. A major drawback of the time-marching approaches is that some severe numerical instability issue is often introduced because of local non-linearity of the governing equations, other instability sources included in the system to be computed or both. Even though fully implicit-type methods are employed, the size of a time step may still have to be relatively small compared to the temporal resolution needed by physicists or engineers due to the requirement constrained by the convergence radius of a nonlinear iterative method. Hence, this situation requires an unacceptable large number of time steps and enormous computational resource. On the other hand, a coupled space-time discretization method proposed by Saitoh et al. [26] is formulated its numerical scheme differently. In this approach, the temporal domain is treated as one additional spatial coordinates, and then the system of PDEs is discretized on a so-called space-time coordinate domain. One notable feature of the space-time approach is that the numerical solution can be obtained in the entire space-time domain simultaneously. Hence, it can produce a considerable reduction of sequential nature compared with that of the conventional time-marching methods. More remarkably, no error propagation or accumulation will occur in the space-time domains. Therefore, it is unconditionally stable and the time step size can be selected arbitrarily. This approach has already been shown numerically to be efficient for a 2D melting/freezing problem [28] and 3D thermal convective flows at high Rayleigh numbers [30], and it was confirmed that the time step is determined only to satisfy the degree of accuracy or the degree of resolution in the timewise direction. Hence, benefiting from the high degree of parallelism, the space-time method attracts more attention in recent years and has been successfully used in many applications, such as PDE-constrained optimization problems [2,39], inverse problems [9,10], the transport problem in a porous medium [14], the fractional Fokker-Planck equation [41], and so on. Besides, Li and Cai [21] developed an optimal convergence theory of the two-level space-time additive Schwarz method for parabolic PDE problems in the space-time formulation, i.e., the convergence of the method is independent of the number of subdomains and the grid size.

The major computational cost of the new approach is on the solution of a large sparse system of nonlinear equations in a fully coupled manner. The robustness of a nonlinear iterative method in the sense that the method can converge for a wide range of physical and system parameters is more demanding than the one needed for the time marching method. Among all available nonlinear iterative solvers, the inexact Newton method (IN) could be a good candidate as the nonlinear solver, since it is easy to implement and