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Inverse Obstacle Scattering in an Unbounded Structure

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Abstract. This paper is concerned with the acoustic scattering of a point incident wave by a sound hard obstacle embedded in a two-layered lossy background medium which is separated by an infinite rough surface. Given the point incident wave, the direct scattering problem is to determine the acoustic wave field for the given obstacle and infinite rough surface; the inverse scattering problem is to determine both the obstacle and the infinite rough surface from the reflected and transmitted wave fields measured on two plane surfaces enclosing the structure. For the direct scattering problem, the well-posedness is studied by using the method of boundary integral equations. For the inverse scattering problem, we prove that the obstacle and the infinite rough surface can be uniquely determined by the measured wave fields corresponding to a single point incident wave. To prove the local stability, the domain derivative of the wave field with respect to the change of the shapes of the obstacle and the infinite rough surface is examined. The local stability indicates that the Hausdorff distance of two domains is bounded above by the distance of corresponding wave fields if the two domains are close enough.

AMS subject classifications: 78A46, 78M30

Key words: Helmholtz's equation, inverse scattering problem, unbounded rough surface, domain derivative, uniqueness, local stability.

1 Introduction

This paper is concerned with the scattering of a point incident wave by an obstacle embedded in a two-layered background medium which is separated by an infinite rough surface. An obstacle is an impenetrable object which has a compact support; an infinite

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rough surface is referred to as a nonlocal perturbation of an infinite plane surface such that the whole surface lies within a finite distance of the original plane. Given the point incident wave, the direct scattering problem is to determine the wave field for the known composite medium consisting of the obstacle and the infinite rough surface; the inverse scattering problem is to determine simultaneously the obstacle and the infinite rough surface by the measured wave fields. The scattering problems arise in diverse scientific areas, such as modeling the optical scattering from obstacles and interfaces of materials in nano-optics, the acoustic wave propagation over the ground and sea surfaces in remote sensing, and the radar object recognition above the sea surface or detection of underwater mines.

Specifically, we study the scattering of a time-harmonic acoustic wave, which is generated by a point source, by a sound hard obstacle and an infinite rough surface. The space above and below the infinite rough surface is filled with a homogeneous and isotropic energy absorbing material, respectively. The wave propagation is modeled by the two-dimensional Helmholtz equation. The scattering problems in such a composite medium are challenging due to the unbounded nature of the interface and the nonlinearity associated with the inverse problem. These scattering problems have received considerable attention and a large amount of work have been done. The recent developments can be found in [25, 26, 33] on the shape and impedance parameter reconstruction in the inverse obstacle scattering problems. The characterization of domain derivatives with respect to local boundary disturbance were studied by Hettlich [20], Hiptmair and Li [22], Kirsch [23], Li [28], Potthast [38]. However, few results are available for rigorous analysis of the obstacle scattering problem in unbounded structures. Our goal is to examine mathematically both the direct and the inverse scattering problems in such a setting.

It is worth mentioning that considerable results are available for the scattering problems in unbounded structures. For unbounded periodic surfaces, many mathematical analysis and numerical computation have been done for both the direct and inverse scattering problems, see Bao [10], Bao, Cui and Li [11], Nédélec and Starling [37], Bao et al. [5] and references cited therein. For the local rough surface scattering problems, Li [28] considered an inverse cavity problem for Maxwell's equations, and showed the global uniqueness and a local stability to reconstruct the cavity wall. Ammari et al. [1] studied the method of integral equations for the electromagnetic scattering from open cavities. An optimization method was introduced in [3] to recover a local rough surface. In [4], a continuation approach over the wave frequency was developed for reconstructing a local rough surface. We refer to Bao et al. [2], Bao and Sun [6], Bao and Lai [9], Li et al. [30], Kress and Tran [24], Zhang and Zhang [42] for various mathematical and numerical methods to solve the local rough surface scattering problems. For the general unbounded non-periodic rough surfaces, the usual Sommerfeld or Silver-Müller radiation condition is not valid any more. Some appropriate radiation condition needs to be given as a part of the boundary value problem. In [16, 17, 41], Chandler-Wilde and Zhang proposed an upward going radiation condition to replace the usual Sommerfeld radiation condition for the scattering problem by rough surfaces and inhomogeneous layers. The well-posedness was established by Chandler-Wilde and Monk [18] for the two-dimensional Helmholtz equation by using a variational approach. Chandler-Wilde et al. [19] studied the well-posedness for the three-dimensional rough surface scattering problem by the method of boundary integral equations. By introducing a transparent boundary condition, Li et al. [27, 31] proved the existence and uniqueness of the weak solution for the electromagnetic scattering problem in an unbounded structure. Lu and Zhang [35] studied the direct and inverse scattering problem by an unbounded rough interface with buried obstacles. When the scattering profile is a sufficiently small and smooth deformation of a plane surface, the analytical solution was introduced in [29] by using a boundary perturbation technique combined with the transformed field expansion. He et al. proposed a spectral method in [21] to solve the unbounded rough surface scattering problem. Zhang et al. [43] developed a regularized conjugate gradient method with fast multipole acceleration for a fractal rough surface scattering problem. We refer to [7, 8, 12, 13, 32, 34, 40] for some mathematical and numerical studies on related inverse scattering problems.

In this work, we introduce an integral radiation condition for the direct scattering problem. The asymptotic behaviour of Green's function is presented. Based on some energy estimates, the uniqueness of the solution is established for the direct scattering problem. The method of boundary integral equations is employed to address the existence of the direct scattering problem. The direct approach is presented to drive the system of boundary integral equations and its well-posedness is discussed. For the inverse problem, we prove that the obstacle and the infinite rough surface are uniquely determined by the reflected and transmitted wave fields measured on the plane surfaces which are above and below the infinite rough surface, respectively. The proof is based on a combination of the Holmgren uniqueness and unique continuation. Based on the well-posedness arguments for the direct scattering problem, we obtain the domain derivative of the wave field with respect to the change of the shapes of the obstacle and the infinite rough surface. Moreover, a local stability is established. It indicates that the Hausdorff distance of two domains, which are characterized by two different obstacles and infinite rough surfaces, is bounded above by the distance of corresponding wave fields if the two obstacles and infinite rough surfaces are close enough.

The outline of the paper is as follows. In Section 2, the model problem is introduced and some asymptotic properties are given for the Green function. Section 3 addresses the solution of the direct scattering problem. The uniqueness is proved. The boundary integral equations are presented and the well-posedness is discussed. The inverse problem is studied in Sections 4 and 5. The global uniqueness of the inverse problem is obtained in Section 4. Section 5 is devoted to the study of the local stability. In particular, the existence and characterization of the domain derivative are examined. As a consequence of the domain derivative, a local stability result is established. The paper is concluded with some general remarks in Section 6.

2 Problem formulation

For $Q\subset\mathbb{R}^2$, denote by $BC(Q)$ the set of bounded and continuous functions on Q. It is a Banach space under the norm

$$\|\phi\|_\infty:=\sup_{x\in Q}|\phi(x)|.$$

For $0<\alpha\leq 1$, denote by $C^{0,\alpha}(Q)$ the Banach space of functions $\phi\in BC(Q)$, which are Hölder continuous with the exponent α. The norm is defined by

$$\|\phi\|_{C^{0,\alpha}(Q)}:=\|\phi\|_\infty+\sup_{\substack{x,y\in Q\\ x\neq y}}\frac{|\phi(x)-\phi(y)|}{|x-y|^\alpha}.$$

Let $C^{1,\alpha}(Q)=\{\phi\in BC(Q)\cap C^1(Q):\nabla\phi\in C^{0,\alpha}(Q)^2\}$, which is a Banach space under the norm

$$\|\phi\|_{C^{1,\alpha}(Q)}:=\|\phi\|_\infty+\|\nabla\phi\|_{C^{0,\alpha}(Q)^2}.$$

Let ν_Γ be the unit normal vector on Γ directed into the exterior of D and let ν_S be the unit normal vector on the boundary S pointing from region Ω_2 to region Ω_1. Define

$$V_{\Gamma_\tau}=\nu_\Gamma\times(V\times\nu_\Gamma)\quad\text{or}\quad V_{S_\tau}=\nu_S\times(V\times\nu_S)$$

and

$$V_{\Gamma_\nu}=\nu_\Gamma\cdot V\quad\text{or}\quad V_{S_\nu}=\nu_S\cdot V,$$

which are the tangential and the normal components of the vector V on Γ or S. Any vector V can be decomposed into its tangential and normal components

$$V|_\Gamma=V_{\Gamma_\tau}+V_{\Gamma_\nu}\nu_\Gamma,\quad V|_S=V_{S_\tau}+V_{S_\nu}\nu_S.$$

Denote by ∇_{Γ_τ} and ∇_{S_τ} the surface gradient on Γ and S, respectively. It is clear to note that

$$\nabla v|_\Gamma=\nabla_{\Gamma_\tau}v+(\nu_\Gamma\cdot\nabla v)\nu_\Gamma,\quad \nabla v|_S=\nabla_{S_\tau}v+(\nu_S\cdot\nabla v)\nu_S,$$

if v is a smooth scalar function defined in a neighborhood of $\Gamma\cup S$.

Now let us specify the problem geometry, which is shown in Fig. 1. Let S be an infinite rough surface which can be described by

$$S=\{x=(x_1,x_2)\in\mathbb{R}^2:x_2=f(x_1)\},$$

where $f\in C^2(\mathbb{R})$. Let

$$f_-=\inf_{x_1\in\mathbb{R}}f(x_1),\quad f_+=\sup_{x_1\in\mathbb{R}}f(x_1).$$

We assume that

$$-\infty<h_-\leq f_-<f_+\leq h_+<\infty,$$

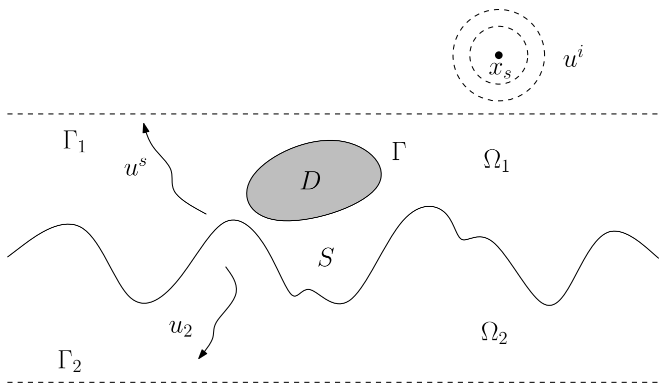

Figure 1: Schematic of the scattering of a point source by an obstacle embedded in a two-layered background medium which is separated by an infinite rough surface.

where h_- and h_+ are two constants. Hence the surface S divides the whole space $\mathbb{R}^2$ into the upper half space Ω_1^+ and the lower half space Ω_2, where

$$\Omega_1^+=\{x\in\mathbb{R}^2: x_2>f(x_1)\},\quad \Omega_2=\{x\in\mathbb{R}^2: x_2<f(x_1)\}.$$

Let D be a bounded obstacle with C^2 boundary Γ. Without loss of generality, the obstacle is assumed to be immersed in the upper half space Ω_1^+, i.e., $D\subset\subset\Omega_1^+$. Define $\Omega_1=\Omega_1^+\setminus\overline{D}$. The results will be same for the case when $D\subset\subset\Omega_2$. Let $\Gamma_j=\{x\in\mathbb{R}^2: x_2=h_j\}, j=1,2$ be the plane surface above the obstacle and below the infinite rough surface, respectively, where the constants h_1,h_2 satisfy $h_2<h_-<h_+<h_1$. Define $R_1=\{x\in\mathbb{R}^2: f(x_1)<x_2<h_1\}$ and $R_2=\{x\in\mathbb{R}^2: h_2<x_2<f(x_1)\}$. Let $R=R_1\cup R_2\cup S$.

For a homogeneous medium in the region $\Omega_j, j=1,2$, the Green function is defined by the solution of the following equation

$$\Delta G_j(x,y)+k_j^2 G_j(x,y)=-\delta(x-y),\quad x\in\Omega_j, \tag{2.1}$$

where k_j is the wavenumber in Ω_j and is assumed to satisfy

$$\Re k_j^2>0,\quad \Im k_j^2>0,\quad \Im k_j>0,$$

which accounts for energy absorbing media. Explicitly, we have

$$G_j(x,y)=\frac{\mathrm{i}}{4}H_0^{(1)}(k_j|x-y|), \tag{2.2}$$

where $H_0^{(1)}$ is the Hankel function of the first kind with order zero.

Let u^i be a point incident wave located at $x_s\in R_1^+=\{x\in\mathbb{R}^2: x_2>h_1\}$. It is given by

$$u^i(x)=G_1(x,x_s)\quad \text{in } \Omega_1. \tag{2.3}$$

It follows from (2.1) that the incident wave satisfies

$$\Delta u^i(x)+k_1^2u^i(x)=-\delta(x-x_s) \quad \text{in } \Omega_1. \tag{2.4}$$

When the incident wave impinges the infinite rough surface S and the obstacle D, the scattered wave u^s and the transmitted wave u_2 will be generated in Ω_1 and Ω_2, respectively. Let $u_1=u^i+u^s$. It can be verified that the total field u_j satisfies

$$\Delta u_1(x)+k_1^2u_1(x)=-\delta(x-x_s), \quad x\in\Omega_1 \tag{2.5}$$

and

$$\Delta u_2(x)+k_2^2u_2(x)=0, \quad x\in\Omega_2. \tag{2.6}$$

The obstacle is assumed to be a sound hard, i.e.,

$$\frac{\partial u_1}{\partial \nu_\Gamma}=0 \quad \text{on } \Gamma. \tag{2.7}$$

The continuity conditions require that u_1 and u_2 satisfy

$$u_1=u_2, \quad \frac{\partial u_1}{\partial \nu_S}=\frac{\partial u_2}{\partial \nu_S} \quad \text{on } S. \tag{2.8}$$

Here

$$\frac{\partial u_1}{\partial \nu_\Gamma}(x):=\lim_{\sigma\to 0^+}\nu_\Gamma(x)\cdot\nabla u_1(x+\sigma\nu_\Gamma(x)), \quad x\in\Gamma$$

and

$$\frac{\partial u_1}{\partial \nu_S}(x):=\lim_{\sigma\to 0^+}\nu_S(x)\cdot\nabla u_1(x+\sigma\nu_S(x)), \quad \frac{\partial u_2}{\partial \nu_S}(x):=\lim_{\sigma\to 0^+}\nu_S(x)\cdot\nabla u_2(x-\sigma\nu_S(x)), \quad x\in S.$$

It was shown in [29] that a transparent boundary condition can be imposed on Γ_j, $j=1,2$:

$$\frac{\partial u_j}{\partial \nu_{\Gamma_j}}=T_ju_j+\rho_j \quad \text{on } \Gamma_j, \tag{2.9}$$

where ν_{Γ_j} is the unit normal vector on Γ_j, T_j is the boundary operator given by

$$(T_ju)(x_1,h_j)=\int_{\mathbb{R}}\mathrm{i}\beta_j(\xi)\hat{u}(\xi,h_j)e^{\mathrm{i}\xi x_1}\mathrm{d}\xi,$$

with

$$\beta_j^2(\xi)=k_j^2-\xi^2, \quad \Im\beta_j(\xi)>0$$

and

$$\rho_1=\frac{\partial u^i}{\partial \nu_{\Gamma_j}}-T_1u^i, \quad \rho_2=0.$$

An appropriate radiation condition is needed since the problem is imposed in the open space $\mathbb{R}^2\backslash\overline{D}$. Due to the infinite extend of the surface S, the usual Sommerfeld radiation condition is not valid. We propose an integral radiation condition which corresponds to the lossy media where $\Im k_j>0$, $j=1,2$:

$$\lim_{r\to+\infty}\int_{\partial B_r^+}\left(|u^s|^2+\left|\frac{\partial u^s}{\partial\nu}\right|^2\right)\mathrm{d}s=0,\quad \lim_{r\to+\infty}\int_{\partial B_r^-}\left(|u_2|^2+\left|\frac{\partial u_2}{\partial\nu}\right|^2\right)\mathrm{d}s=0, \tag{2.10}$$

where ν denote the unit normal vector on the boundary ∂B_r directed into the exterior of $B_r(O)$. Here $B_r(O)$ is a ball centered at the origin with radius r and boundary $\partial B_r=\partial B_r^+\cup\partial B_r^-$, where ∂B_r^+ and ∂B_r^- denote the semi-circle above and below S, respectively.

Given the point incident field u^i in (2.3), the direct scattering problem is to find u_1 and u_2 which satisfy (2.5)-(2.8) and (2.10); the inverse scattering problem is to determine simultaneously the boundary of the obstacle Γ and the infinite rough surface S from the measured wave field u_j on Γ_j, $j=1,2$. We study both of the direct and inverse scattering problems.

Denote by $\mathcal{T}_j$ the set of functions $\phi\in C^2(\Omega_j)\cap C^{1,\alpha}(\overline{\Omega}_j)$, $j=1,2$. More specifically, the scattering problem can be stated as follows.

Problem 2.1. Given the incident field u^i in (2.3), the direct scattering problem is to determine $u^s\in\mathcal{T}_1$ and $u_2\in\mathcal{T}_2$ such that

(i) The total fields $u_1=u^s+u^i$ and u_2 satisfy (2.5) and (2.6), respectively;

(ii) The total field u_1 satisfies the sound hard boundary condition (2.7);

(iii) The total fields u_1 and u_2 satisfy the transmission boundary conditions (2.8);

(iv) The scattered fields u^s and the transmitted fields u_2 satisfy the radiation conditions (2.10).

The following result gives the asymptotic behaviour of the Green function and plays an important role when formulating the boundary integral equations for the direct scattering problem.

Lemma 2.1. *For each fixed $y\in\Omega_j$, $j=1,2$, the Green function G_j admits the asymptotic behaviour*

$$|G_j(x,y)|,\ \left|\frac{\partial G_j(x,y)}{\partial\nu(y)}\right|,\ \left|\frac{\partial^2 G_j(x,y)}{\partial\nu(x)\partial\nu(y)}\right|\le C\left(\frac{\exp(-\frac{1}{2}\Im(k_j)|x|)}{|x|^{\frac{1}{2}}}\right),\quad |x|\to\infty,$$

where C is a constant independent of x and y.

Proof. A simple calculation yields

$$\frac{\partial G_j(x,y)}{\partial x_l}=-\frac{\partial G_j(x,y)}{\partial y_l}=-\frac{\mathrm{i}k_j(x_l-y_l)}{4|x-y|}H_1^{(1)}(k_j|x-y|),\quad l=1,2,$$

and

$$\frac{\partial^2 G_j(x,y)}{\partial x_m \partial y_l} = \frac{\mathrm{i}k_j}{4}\left[\frac{k_j(x_m-y_m)(x_l-y_l)}{|x-y|^2}H_0^{(1)}(k_j|x-y|)+\frac{\delta_{ml}}{|x-y|}H_1^{(1)}(k_j|x-y|)\right.$$
$$\left. -2\frac{(x_m-y_m)(x_l-y_l)}{|x-y|^3}H_1^{(1)}(k_j|x-y|)\right], \quad l,m=1,2,$$

where $H_1^{(1)}$ is the Hankel function of the first kind with order one and δ_{ml} is the Kronecker delta function. Using the asymptotic expansion

$$H_\mu^{(1)}(k_j|x-y|) \simeq \sqrt{\frac{2}{\pi k_j|x-y|}}\exp\left(\mathrm{i}\left(k_j|x-y|-\mu\frac{\pi}{2}-\frac{\pi}{4}\right)\right)$$
$$=\frac{2\exp\left(-\mathrm{i}(\mu\frac{\pi}{2}+\frac{\pi}{4})\right)}{\sqrt{2\pi k_j}}\left[\frac{\exp(\mathrm{i}k_j|x-y|)}{|x-y|^{\frac{1}{2}}}\right], \quad |x-y|\to+\infty, \quad \mu=0,1,$$

we have

$$|G_j(x,y)|, \left|\frac{\partial G_j(x,y)}{\partial \nu(y)}\right|, \left|\frac{\partial^2 G_j(x,y)}{\partial \nu(x)\partial \nu(y)}\right| \le C_1\left[\frac{\exp(-\Im(k_j)|x-y|)}{|x-y|^{\frac{1}{2}}}\right], \quad |x-y|\to+\infty, \tag{2.11}$$

where C_1 is a constant independent of x and y.

For each fixed $y\in\Omega_j$, $j=1,2$, we have $\hat{x}\cdot y<\frac{|x|}{2}$ for $|x|\to\infty$, where $\hat{x}=x/|x|$. Since

$$|x-y|=\sqrt{|x|^2-2x\cdot y+|y|^2}=|x|-\hat{x}\cdot y+\mathcal{O}\left(\frac{1}{|x|}\right)\ge\frac{|x|}{2}+\mathcal{O}\left(\frac{1}{|x|}\right), \quad |x|\to\infty, \tag{2.12}$$

we conclude from $\Im(k_j)>0$ and (2.11)-(2.12) that there exists a constant C independent of x,y such that

$$|G_j(x,y)|, \left|\frac{\partial G_j(x,y)}{\partial \nu(y)}\right|, \left|\frac{\partial^2 G_j(x,y)}{\partial \nu(x)\partial \nu(y)}\right| \le C\left(\frac{\exp(-\frac{1}{2}\Im(k_j)|x|)}{|x|^{\frac{1}{2}}}\right), \quad |x|\to\infty,$$

which completes the proof. □

3 The direct scattering problem

In this section, we discuss the well-posedness of the direct scattering problem. First, the problem is shown to have a unique solution. The method of boundary integral equations is used to address the existence of the solution.

3.1 Uniqueness

Theorem 3.1. *The direct scattering problem has at most one solution.*

Proof. Suppose that (v_1,v_2) and $(\tilde{v}_1,\tilde{v}_2)$ are two solutions of Problem 2.1. Let $u_1=v_1-\tilde{v}_1$ and $u_2=v_2-\tilde{v}_2$, then u_j, $j=1,2$ satisfy the homogeneous Helmholtz equation $\Delta u_j+k_j^2u_j=0$ with conditions (2.7)-(2.8) and (2.10).

Denote $\Omega_r=(B_r\cap\Omega_1)$ with boundary $\partial\Omega_r=\partial B_r^+\cup\Gamma\cup S_r$, where $\partial B_r^+=\partial B_r\cap\Omega_1$ and $S_r=S\cap B_r$. For each fixed $x\in\Omega_r$, applying the Green first theorem to u_1 in Ω_r, we have

$$\int_{\Omega_r}\left[|\nabla u_1|^2+u_1(\Delta\overline{u}_1)\right]\mathrm{d}x=\int_{\partial\Omega_r}u_1\frac{\partial\overline{u}_1}{\partial\nu}\mathrm{d}s_x=\left(\int_{\partial B_r^+}+\int_\Gamma+\int_{S_r}\right)u_1\frac{\partial\overline{u}_1}{\partial\nu}\mathrm{d}s_x, \tag{3.1}$$

where $\nu=\nu(x)$ stands for the unit normal vector at $x\in\partial\Omega_r$ pointing towards the exterior of Ω_r. Letting $r\to+\infty$, we have from (2.7), (2.8), (2.10) and (3.1) that

$$\begin{aligned}\int_{\Omega_1}\left[|\nabla u_1|^2+u_1(\Delta\overline{u}_1)\right]\mathrm{d}x&=\int_S u_1\frac{\partial\overline{u}_1}{\partial\nu}\mathrm{d}s_x+\int_\Gamma u_1\frac{\partial\overline{u}_1}{\partial\nu}\mathrm{d}s_x\\&=-\int_S u_1\frac{\partial\overline{u}_1}{\partial\nu_S}\mathrm{d}s_x-\int_\Gamma u_1\frac{\partial\overline{u}_1}{\partial\nu_\Gamma}\mathrm{d}s_x\\&=-\int_S u_1\frac{\partial\overline{u}_1}{\partial\nu_S}\mathrm{d}s_x,\end{aligned} \tag{3.2}$$

where ν_Γ denotes the unit normal vector on the boundary Γ directed into the exterior of D, $\nu_S(x)$ denotes the unit normal vector at $x\in S$ pointing from region Ω_2 to region Ω_1.

Using (3.2) and the homogeneous Helmholtz equation $\Delta u_1+k_1^2u_1=0$, yields

$$\begin{aligned}\int_{\Omega_1}\left[|\nabla u_1|^2+u_1(\Delta\overline{u}_1)\right]\mathrm{d}x&=\int_{\Omega_1}\left(|\nabla u_1|^2-\overline{k_1^2}|u_1|^2\right)\mathrm{d}x\\&=\int_{\Omega_1}\left(|\nabla u_1|^2-\Re(k_1^2)|u_1|^2+\mathrm{i}\Im(k_1^2)|u_1|^2\right)\mathrm{d}x\\&=-\int_S u_1\frac{\partial\overline{u}_1}{\partial\nu_S}\mathrm{d}s_x,\end{aligned} \tag{3.3}$$

which gives by taking the imaginary part of (3.3) that

$$-\Im\left[\int_S u_1\frac{\partial\overline{u}_1}{\partial\nu_S}\mathrm{d}s_x\right]=\Im(k_1^2)\int_{\Omega_1}|u_1|^2\mathrm{d}x\geq 0. \tag{3.4}$$

Similarly, we obtain

$$\Im\left[\int_S u_2\frac{\partial\overline{u}_2}{\partial\nu_S}\mathrm{d}s_x\right]=\Im(k_2^2)\int_{\Omega_2}|u_2|^2\mathrm{d}x\geq 0. \tag{3.5}$$

Noting the boundary conditions (2.8), we have

$$\int_S u_1\frac{\partial\overline{u}_1}{\partial\nu_S}\mathrm{d}s_x=\int_S u_2\frac{\partial\overline{u}_2}{\partial\nu_S}\mathrm{d}s_x. \tag{3.6}$$

It follows immediately from combining (3.4)-(3.6) that

$$\Im(k_1^2)\int_{\Omega_1}|u_1|^2\mathrm{d}x+\Im(k_2^2)\int_{\Omega_2}|u_2|^2\mathrm{d}x=0. \tag{3.7}$$

With the aid of $\Im(k_j^2)>0$, $j=1,2$, we have from (3.7) that

$$\int_{\Omega_1}|u_1|^2\mathrm{d}x=\int_{\Omega_2}|u_2|^2\mathrm{d}x=0,$$

which implies that $v_1=\tilde{v}_1$ in Ω_1 and $v_2=\tilde{v}_2$ in Ω_2. □

3.2 Potential operators

We introduce several potential operators in order to derive the boundary integral equations for the direct scattering problem.

On the infinite rough surface S, we define the single-layer potential operator $\mathbf{S}_S: C^{0,\alpha}(S)\to C^{1,\alpha}(S)$ and the double-layer potential operator $\mathbf{K}_S: C^{1,\alpha}(S)\to C^{1,\alpha}(S)$ by

$$(\mathbf{S}_S\phi)(x)=\int_S\big(G_1(x,y)-G_2(x,y)\big)\phi(y)\mathrm{d}s_y,\quad x\in S,$$
$$(\mathbf{K}_S\phi)(x)=\int_S\left(\frac{\partial G_1(x,y)}{\partial\nu_S(y)}-\frac{\partial G_2(x,y)}{\partial\nu_S(y)}\right)\phi(y)\mathrm{d}s_y,\quad x\in S,$$

where ϕ is called the density. Define the normal derivative operators $\mathbf{K}_S^*: C^{0,\alpha}(S)\to C^{1,\alpha}(S)$ and $\mathbf{T}_S: C^{1,\alpha}(S)\to C^{0,\alpha}(S)$ by

$$(\mathbf{K}_S^*\phi)(x)=\int_S\left(\frac{\partial G_1(x,y)}{\partial\nu_S(x)}-\frac{\partial G_2(x,y)}{\partial\nu_S(x)}\right)\phi(y)\mathrm{d}s_y,\quad x\in S,$$
$$(\mathbf{T}_S\phi)(x)=\frac{\partial}{\partial\nu_S(x)}\int_S\left(\frac{\partial G_1(x,y)}{\partial\nu_S(y)}-\frac{\partial G_2(x,y)}{\partial\nu_S(y)}\right)\phi(y)\mathrm{d}s_y,\quad x\in S.$$

On Γ, we define the double-layer potential operator $\mathbf{K}_\Gamma: C^{0,\alpha}(\Gamma)\to C^{1,\alpha}(\Gamma)$ by

$$(\mathbf{K}_\Gamma\phi)(x)=2\int_\Gamma\frac{\partial G_1(x,y)}{\partial\nu_\Gamma(y)}\phi(y)\mathrm{d}s_y,\quad x\in\Gamma,$$

and the normal derivative operator $\mathbf{K}_\Gamma^*: C^{0,\alpha}(\Gamma)\to C^{1,\alpha}(\Gamma)$ by

$$(\mathbf{K}_\Gamma^*\phi)(x)=2\int_\Gamma\frac{\partial G_1(x,y)}{\partial\nu_\Gamma(x)}\phi(y)\mathrm{d}s_y,\quad x\in\Gamma.$$

Let $S_A=\{x\in S:|x_1|\le A\}$, where $A>0$ is a constant. Define the truncated single-layer potential operator $\mathbf{S}_A: C^{0,\alpha}(S_A)\to C^{1,\alpha}(S)$ and the truncated double-layer potential operator $\mathbf{K}_A : C^{1,\alpha}(S_A) \to C^{1,\alpha}(S)$ by

$$(\mathbf{S}_A\phi)(x) = \int_{S_A} \big(G_1(x,y) - G_2(x,y)\big)\phi_2(y)\mathrm{d}s_y, \quad x \in S,$$

$$(\mathbf{K}_A\phi)(x) = \int_{S_A} \left(\frac{\partial G_1(x,y)}{\partial \nu_S(y)} - \frac{\partial G_2(x,y)}{\partial \nu_S(y)}\right)\phi_1(y)\mathrm{d}s_y, \quad x \in S.$$

Similarly, we define the truncated normal derivative operators $\mathbf{K}_A^* : C^{0,\alpha}(S_A) \to C^{1,\alpha}(S)$ and $\mathbf{T}_A : C^{1,\alpha}(S_A) \to C^{0,\alpha}(S)$ by

$$(\mathbf{K}_A^*\phi)(x) = \int_{S_A} \left(\frac{\partial G_1(x,y)}{\partial \nu_S(x)} - \frac{\partial G_2(x,y)}{\partial \nu_S(x)}\right)\phi_2(y)\mathrm{d}s_y, \quad x \in S,$$

$$(\mathbf{T}_A\phi)(x) = \int_{S_A} \left(\frac{\partial^2 G_1(x,y)}{\partial \nu_S(x)\partial \nu_S(y)} - \frac{\partial^2 G_2(x,y)}{\partial \nu_S(x)\partial \nu_S(y)}\right)\phi_1(y)\mathrm{d}s_y, \quad x \in S.$$

3.3 Boundary integral equations

In this section, we present the direct approach to derive the boundary integral equations for the direct scattering problem.

Let $B_r = \{x \in \mathbb{R}^2 : |x| < r\}$. Denote $\Omega_r = B_r \cap \Omega_1$ with the boundary $\partial\Omega_r = \partial B_r^+ \cup \Gamma \cup S_r$, where $\partial B_r^+ = \partial B_r \cap \Omega_1$ and $S_r = S \cap B_r$. For each fixed $x \in \Omega_r$, applying the Green second theorem to u_1 and G_1 in the region Ω_r, we obtain

$$\begin{aligned}&\int_{\Omega_r} \big[u_1(y)\Delta G_1(x,y) - G_1(x,y)\Delta u_1(y)\big]\mathrm{d}y \\ &= \int_{\partial\Omega_r} \left[u_1(y)\frac{\partial G_1(x,y)}{\partial \nu(y)} - \frac{\partial u_1(y)}{\partial \nu(y)}G_1(x,y)\right]\mathrm{d}s_y, \quad x \in \Omega_1,\end{aligned} \tag{3.8}$$

where $\nu(y)$ stands for the unit normal vector at $y \in \partial\Omega_r$ pointing towards the exterior of Ω_r.

It follows from (2.1), (2.3), and (2.5) that

$$\begin{aligned}&\int_{\Omega_r} \big[u_1(y)\Delta G_1(x,y) - G_1(x,y)\Delta u_1(y)\big]\mathrm{d}y \\ &= \int_{\Omega_r} u_1(y)[\Delta G_1(x,y) + k_1^2 G_1(x,y)]\mathrm{d}y - \int_{\Omega_r} [\Delta u_1(y) + k_1^2 u_1(y)]G_1(x,y)\mathrm{d}y \\ &= -\int_{\Omega_r} \delta(x-y)u_1(y)\mathrm{d}y - \int_{\Omega_r} [-\delta(y-x_s)]G_1(x,y)\mathrm{d}y \\ &= -u_1(x) + G_1(x,x_s) = -[u_1(x) - u^i(x)].\end{aligned} \tag{3.9}$$

Letting $r \to +\infty$ and using (3.8)-(3.9), we have

$$\begin{aligned}u_1(x) - u^i(x) &= -\lim_{r\to+\infty}\int_{\partial\Omega_r} \left[u_1(y)\frac{\partial G_1(x,y)}{\partial \nu(y)} - \frac{\partial u_1(y)}{\partial \nu(y)}G_1(x,y)\right]\mathrm{d}s_y \\ &= -\left(\int_S + \int_\Gamma + \lim_{r\to+\infty}\int_{\partial B_r^+}\right)\left[u_1(y)\frac{\partial G_1(x,y)}{\partial \nu(y)} - \frac{\partial u_1(y)}{\partial \nu(y)}G_1(x,y)\right]\mathrm{d}s_y.\end{aligned} \tag{3.10}$$

It follows from Lemma 2.1 and the radiation conditions (2.10) that we obtain

$$\begin{aligned}&\left|\int_{\partial B_r^+}\left[u_1(y)\frac{\partial G_1(x,y)}{\partial\nu(y)}-\frac{\partial u_1(y)}{\partial\nu(y)}G_1(x,y)\right]\mathrm{d}s_y\right|\\ \leq&\left[\int_{\partial B_r^+}|u_1(y)|^2\mathrm{d}s_y\right]^{\frac{1}{2}}\left[\int_{\partial B_r^+}\left|\frac{\partial G_1(x,y)}{\partial\nu(y)}\right|^2\mathrm{d}s_y\right]^{\frac{1}{2}}\\ &+\left[\int_{\partial B_r^+}\left|\frac{\partial u_1(y)}{\partial\nu(y)}\right|^2\mathrm{d}s_y\right]^{\frac{1}{2}}\left[\int_{\partial B_r^+}|G_1(x,y)|^2\mathrm{d}s_y\right]^{\frac{1}{2}}\to 0,\quad r\to+\infty. \end{aligned}\tag{3.11}$$

Using (3.10)-(3.11) and the condition (ii) in Problem 2.1, and letting $r\to+\infty$, we have for each fixed $x\in\Omega_1$ that

$$\begin{aligned}&u_1(x)-u^i(x)\\ =&-\int_S\left[u_1(y)\frac{\partial G_1(x,y)}{\partial\nu(y)}-\frac{\partial u_1(y)}{\partial\nu(y)}G_1(x,y)\right]\mathrm{d}s_y-\int_\Gamma\left[u_1(y)\frac{\partial G_1(x,y)}{\partial\nu(y)}-\frac{\partial u_1(y)}{\partial\nu(y)}G_1(x,y)\right]\mathrm{d}s_y\\ =&\int_S\left[u_1(y)\frac{\partial G_1(x,y)}{\partial\nu_S(y)}-\frac{\partial u_1(y)}{\partial\nu_S(y)}G_1(x,y)\right]\mathrm{d}s_y+\int_\Gamma u_1(y)\frac{\partial G_1(x,y)}{\partial\nu_\Gamma(y)}\mathrm{d}s_y,\end{aligned}$$

which gives

$$\begin{aligned}u_1(x)=&u^i(x)+\int_S\left[\frac{\partial G_1(x,y)}{\partial\nu_S(y)}u_1(y)-G_1(x,y)\frac{\partial u_1(y)}{\partial\nu_S(y)}\right]\mathrm{d}s_y\\ &+\int_\Gamma\frac{\partial G_1(x,y)}{\partial\nu_\Gamma(y)}u_1(y)\mathrm{d}s_y,\quad x\in\Omega_1,\end{aligned}\tag{3.12}$$

Noting the continuity conditions (2.8), we may apply the second Green theorem to u_2 and G_2 in Ω_2 and obtain

$$u_2(x)=-\int_S\left[\frac{\partial G_2(x,y)}{\partial\nu_S(y)}u_1(y)-G_2(x,y)\frac{\partial u_1(y)}{\partial\nu_S(y)}\right]\mathrm{d}s_y,\quad x\in\Omega_2.\tag{3.13}$$

It follows from the jump conditions of the single- and double-layer potentials and (3.12) that the field u_1 satisfy the boundary integral equation

$$\begin{aligned}u_1(x)=&2\int_\Gamma\frac{\partial G_1(x,y)}{\partial\nu_\Gamma(y)}u_1(y)\mathrm{d}s_y+2\int_S\left[\frac{\partial G_1(x,y)}{\partial\nu_S(y)}u_1(y)-G_1(x,y)\frac{\partial u_1(y)}{\partial\nu_S(y)}\right]\mathrm{d}s_y\\ &+2u^i(x),\quad x\in\Gamma.\end{aligned}\tag{3.14}$$

In (3.12) and (3.13), using the jump conditions of the single- and double-layer potentials and the continuity condition $u_1|_S=u_2|_S$ in (2.8), we get the boundary integral equation

$$\begin{aligned}u_1(x)=&\int_S\left[\left(\frac{\partial G_1(x,y)}{\partial\nu_S(y)}-\frac{\partial G_2(x,y)}{\partial\nu_S(y)}\right)u_1(y)-\left(G_1(x,y)-G_2(x,y)\right)\frac{\partial u_1(y)}{\partial\nu_S(y)}\right]\mathrm{d}s_y\\ &+\int_\Gamma\frac{\partial G_1(x,y)}{\partial\nu_\Gamma(y)}u_1(y)\mathrm{d}s_y+u^i(x),\quad x\in S.\end{aligned}\tag{3.15}$$

Note that the boundary integral equations (3.14)-(3.15) involve the unknown $\frac{\partial u_1}{\partial \nu_S}$ on S. It requires to take the normal derivatives of (3.12) and (3.13) on S, which leads to boundary integral equations with hyper-singular kernels. We combine the normal derivatives of (3.12) and (3.13) to avoid this issue. Taking the normal derivatives of (3.12) and (3.13) on S and adding them together, using the jump conditions of the single- and double-layer potentials, and noting the jump condition $\frac{\partial u_1}{\partial \nu_S}\big|_S = \frac{\partial u_2}{\partial \nu_S}\big|_S$ in (2.8), we have the boundary integral equation

$$\begin{aligned}\frac{\partial u_1(x)}{\partial \nu_S(x)} =& \int_S \left[\left(\frac{\partial^2 G_1(x,y)}{\partial \nu_S(x)\partial \nu_S(y)} - \frac{\partial^2 G_2(x,y)}{\partial \nu_S(x)\partial \nu_S(y)} \right) u_1(y) - \left(\frac{\partial G_1(x,y)}{\partial \nu_S(x)} - \frac{\partial G_2(x,y)}{\partial \nu_S(x)} \right) \frac{\partial u_1(y)}{\partial \nu_S(y)} \right] \mathrm{d}s_y \\ &+ \int_\Gamma \frac{\partial^2 G_1(x,y)}{\partial \nu_S(x)\partial \nu_\Gamma(y)} u_1(y) \mathrm{d}s_y + \frac{\partial u^i(x)}{\partial \nu_S(x)}, \quad x \in S. \end{aligned} \tag{3.16}$$

Theorem 3.2. *Let* $u^s \in \mathcal{T}_1, u_2 \in \mathcal{T}_2$ *have the integral representations* (3.12)-(3.13) *and satisfy the boundary integral equations* (3.14)-(3.16)*. Then* (u_1,u_2) *are the solutions of Problem* 2.1.

Proof. We only show the proof for the field u_1. The corresponding result can be similarly proved for the field u_2. If the field $u^s \in \mathcal{T}_1$ has the integral representation (3.12), then we have

$$u^s(x) = \int_S \left[\frac{\partial G_1(x,y)}{\partial \nu_S(y)} u_1(y) - G_1(x,y) \frac{\partial u_1(y)}{\partial \nu_S(y)} \right] \mathrm{d}s_y + \int_\Gamma \frac{\partial G_1(x,y)}{\partial \nu_\Gamma(y)} u_1(y) \mathrm{d}s_y, \quad x \in \Omega_1. \tag{3.17}$$

Noting that for any $x \in \Omega_1$ and $y \in S \cup \Gamma$, we have $x \neq y$. Thus, from (2.1) and (3.17), we obtain that

$$\begin{aligned} &\Delta u^s(x) + k_1^2 u^s(x) \\ =& \int_S \left(u_1(y) \frac{\partial}{\partial \nu_S} (\Delta_x G_1(x,y) + k_1^2 G_1(x,y)) - \frac{\partial u_1(y)}{\partial \nu_S} (\Delta_x G_1(x,y) + k_1^2 G_1(x,y)) \right) \mathrm{d}s_y \\ &+ \int_\Gamma u_1(y) \frac{\partial}{\partial \nu_\Gamma} (\Delta_x G_1(x,y) + k_1^2 G_1(x,y)) \mathrm{d}s_y = 0, \quad x \in \Omega_1. \end{aligned} \tag{3.18}$$

It follows from (2.4) and (3.18) that

$$\Delta u_1 + k_1^2 u_1 = (\Delta u^s + k_1^2 u^s) + (\Delta u^i + k_1^2 u^i) = -\delta(x - x_s) \quad \text{in } \Omega_1. \tag{3.19}$$

Furthermore, with the help of Lemma 2.1 and (3.17), we deduce that

$$\begin{aligned} |u^s(x)| \le C \bigg[&\int_S |G_1(x,y)| \left| \frac{\partial u_1(y)}{\partial \nu_S(y)} \right| \mathrm{d}s_y + \int_S \left| \frac{\partial G_1(x,y)}{\partial \nu_S(y)} \right| |u_1(y)| \mathrm{d}s_y \\ &+ \int_\Gamma |u_1(y)| \left| \frac{\partial G_1(x,y)}{\partial \nu_\Gamma(y)} \right| \mathrm{d}s_y \bigg] \end{aligned}$$

$$\begin{aligned}&\leq C\left[\left\|\frac{\partial u_1}{\partial \nu_S}\right\|_{C^{0,\alpha}(S)}\int_S |G_1(x,y)|\mathrm{d}s_y+\|u_1\|_{C^{0,\alpha}(S)}\int_S\left|\frac{\partial G_1(x,y)}{\partial \nu_S(y)}\right|\mathrm{d}s_y\right.\\&\qquad\left.+\|u_1\|_{C^{0,\alpha}(\Gamma)}\int_\Gamma\left|\frac{\partial G_1(x,y)}{\partial \nu_\Gamma(y)}\right|\mathrm{d}s_y\right]\\&\leq C\left[\lim_{r\to+\infty}\int_{S_r}|G_1(x,y)|\mathrm{d}s_y+\lim_{r\to+\infty}\int_{S_r}\left|\frac{\partial G_1(x,y)}{\partial \nu_S(y)}\right|\mathrm{d}s_y\right.\\&\qquad\left.+\int_\Gamma\left|\frac{\partial G_1(x,y)}{\partial \nu_\Gamma(y)}\right|\mathrm{d}s_y\right].\end{aligned}\tag{3.20}$$

For each fixed $r\geq 1$, as $|x|\to+\infty$, by Lemma 2.1, we have

$$\begin{aligned}\int_{S_r}|G_1(x,y)|\mathrm{d}s_y&\leq C\int_{S_r}\left|\frac{\exp(\frac{1}{2}\mathrm{i}k_1|x-y|)}{|x-y|^{\frac{1}{2}}}\exp\left(\frac{1}{2}\mathrm{i}k_1|x-y|\right)\right|\mathrm{d}s_y\\&\leq C\frac{\exp(-\frac{1}{4}\Im(k_1)|x|)}{|x|^{\frac{1}{2}}}\int_0^r\exp\left(-\frac{1}{2}\Im(k_1)y_1\right)\mathrm{d}y_1\\&\leq C\frac{\exp(-\frac{1}{4}\Im(k_1)|x|)}{|x|^{\frac{1}{2}}}\left(1-\exp\left(-\frac{r}{2}\Im(k_1)\right)\right).\end{aligned}\tag{3.21}$$

Similarly, we may show that

$$\int_{S_r}\left|\frac{\partial G_1(x,y)}{\partial \nu_S(y)}\right|\mathrm{d}s_y\leq C\frac{\exp(-\frac{1}{4}\Im(k_1)|x|)}{|x|^{\frac{1}{2}}}\left(1-\exp\left(-\frac{r}{2}\Im(k_1)\right)\right),\tag{3.22}$$

$$\int_{\Gamma}\left|\frac{\partial G_1(x,y)}{\partial \nu_\Gamma(y)}\right|\mathrm{d}s_y\leq C\frac{\exp(-\frac{1}{2}\Im(k_1)|x|)}{|x|^{\frac{1}{2}}}.\tag{3.23}$$

Combining (3.20)-(3.23) and noting $\Im(k_1)>0$, we obtain

$$|u^s(x)|\leq C\left(\frac{\exp(-\frac{1}{4}\Im(k_1)|x|)}{|x|^{\frac{1}{2}}}\right),\quad |x|\to+\infty$$

and

$$\begin{aligned}\int_{\partial B_r^+}|u^s|^2\mathrm{d}s_x&\leq C\int_{\partial B_r^+}\frac{\exp(-\frac{1}{2}\Im(k_1)|x|)}{|x|}\mathrm{d}s_x\\&\leq C\left(\frac{\exp(-\frac{1}{2}\Im(k_1)r)}{r}2\pi r\right)\leq C\exp\left(-\frac{1}{2}\Im(k_1)r\right)\to 0,\quad r\to+\infty,\end{aligned}$$

where C is a positive constant independent of r.

Similarly, we can show that

$$\lim_{r\to+\infty}\int_{\partial B_r^+}\left|\frac{\partial u^s(x)}{\partial \nu(x)}\right|^2\mathrm{d}s_x=\lim_{r\to+\infty}\int_{\partial B_r^-}|u_2|^2\mathrm{d}s_x=\lim_{r\to+\infty}\int_{\partial B_r^-}\left|\frac{\partial u_2(x)}{\partial \nu(x)}\right|^2\mathrm{d}s_x=0.$$

Since u_1 satisfies (3.19) and the above radiation condition, applying the Green second theorem to u_1 and G_1 in the region Ω_1, using the jump conditions of the single- and double-layer potentials, we obtain the boundary integral equation

$$u_1(x)=2\int_S\left[\frac{\partial G_1(x,y)}{\partial\nu_S(y)}u_1(y)-G_1(x,y)\frac{\partial u_1(y)}{\partial\nu_S(y)}\right]\mathrm{d}s_y \\ +2\int_\Gamma\left[\frac{\partial G_1(x,y)}{\partial\nu_\Gamma(y)}u_1(y)-G_1(x,y)\frac{\partial u_1(y)}{\partial\nu_\Gamma(y)}\right]\mathrm{d}s_y+2u^i(x),\quad x\in\Gamma. \tag{3.24}$$

From (3.14) and (3.24), it is easy to verify that $\frac{\partial u_1}{\partial\nu_\Gamma}|_\Gamma=0$.

In (3.12) and (3.13), using the jump conditions of the single- and double-layer potentials, we get the boundary integral equations

$$\frac{1}{2}u_1(x)=u^i(x)+\int_S\left[\frac{\partial G_1(x,y)}{\partial\nu_S(y)}u_1(y)-G_1(x,y)\frac{\partial u_1(y)}{\partial\nu_S(y)}\right]\mathrm{d}s_y \\ +\int_\Gamma\frac{\partial G_1(x,y)}{\partial\nu_\Gamma(y)}u_1(y)\mathrm{d}s_y,\quad x\in S, \tag{3.25}$$

and

$$u_2(x)-\frac{1}{2}u_1(x)=-\int_S\left[\frac{\partial G_2(x,y)}{\partial\nu_S(y)}u_1(y)-G_2(x,y)\frac{\partial u_1(y)}{\partial\nu_S(y)}\right]\mathrm{d}s_y,\quad x\in S. \tag{3.26}$$

Now adding (3.25) and (3.26) gives

$$u_2(x)=\int_S\left[\left(\frac{\partial G_1(x,y)}{\partial\nu_S(y)}-\frac{\partial G_2(x,y)}{\partial\nu_S(y)}\right)u_1(y)-\big(G_1(x,y)-G_2(x,y)\big)\frac{\partial u_1(y)}{\partial\nu_S(y)}\right]\mathrm{d}s_y \\ +\int_\Gamma\frac{\partial G_1(x,y)}{\partial\nu_\Gamma(y)}u_1(y)\mathrm{d}s_y+u^i(x),\quad x\in S. \tag{3.27}$$

From (3.15) and (3.27), it is easy to verify that $u_1|_S=u_2|_S$.

Taking the normal derivatives of (3.12) and (3.13) on S, using the jump conditions of the single- and double-layer potentials, we get the boundary integral equations

$$\frac{1}{2}\frac{\partial u_1(x)}{\partial\nu_S(x)}=\frac{\partial u^i(x)}{\partial\nu_S(x)}+\int_S\left[\frac{\partial^2 G_1(x,y)}{\partial\nu_S(x)\partial\nu_S(y)}u_1(y)-\frac{\partial G_1(x,y)}{\partial\nu_S(x)}\frac{\partial u_1(y)}{\partial\nu_S(y)}\right]\mathrm{d}s_y \\ +\int_\Gamma\frac{\partial^2 G_1(x,y)}{\partial\nu_S(x)\partial\nu_\Gamma(y)}u_1(y)\mathrm{d}s_y,\quad x\in S, \tag{3.28}$$

and

$$\frac{\partial u_2(x)}{\partial\nu_S(x)}-\frac{1}{2}\frac{\partial u_1(x)}{\partial\nu_S(x)}=-\int_S\left[\frac{\partial^2 G_2(x,y)}{\partial\nu_S(x)\partial\nu_S(y)}u_1(y)-\frac{\partial G_2(x,y)}{\partial\nu_S(x)}\frac{\partial u_1(y)}{\partial\nu_S(y)}\right]\mathrm{d}s_y,\quad x\in S. \tag{3.29}$$

Adding them together, we have the boundary integral equation

$$\frac{\partial u_2(x)}{\partial \nu_S(x)} = \int_S \left[\left(\frac{\partial^2 G_1(x,y)}{\partial \nu_S(x)\partial \nu_S(y)} - \frac{\partial^2 G_2(x,y)}{\partial \nu_S(x)\partial \nu_S(y)} \right) u_1(y) - \left(\frac{\partial G_1(x,y)}{\partial \nu_S(x)} - \frac{\partial G_2(x,y)}{\partial \nu_S(x)} \right) \frac{\partial u_1(y)}{\partial \nu_S(y)} \right] \mathrm{d}s_y$$
$$+ \int_\Gamma \frac{\partial^2 G_1(x,y)}{\partial \nu_S(x)\partial \nu_\Gamma(y)} u_1(y)\mathrm{d}s_y + \frac{\partial u^i(x)}{\partial \nu_S(x)}, \quad x \in S. \tag{3.30}$$

From (3.16) and (3.30), it is easy to verify that $\frac{\partial u_1}{\partial \nu_S}\big|_S = \frac{\partial u_2}{\partial \nu_S}\big|_S$. □

The system of boundary integral equations (3.14)-(3.16) can be written in the operator form

$$\begin{bmatrix} \mathbf{I}-\mathbf{K}_\Gamma & -\mathbf{K}_{S,\Gamma} & \mathbf{S}_{S,\Gamma} \\ -\mathbf{K}_{\Gamma,S} & \mathbf{I}-\mathbf{K}_S & \mathbf{S}_S \\ -\mathbf{T}_{\Gamma,S} & -\mathbf{T}_S & \mathbf{I}+\mathbf{K}_S^* \end{bmatrix} \begin{bmatrix} u_1|_\Gamma \\ u_1|_S \\ \frac{\partial u_1}{\partial \nu_S}\big|_S \end{bmatrix} = \begin{bmatrix} 2u^i|_\Gamma \\ u^i|_S \\ \frac{\partial u^i}{\partial \nu_S}\big|_S \end{bmatrix}, \tag{3.31}$$

where $\mathbf{I}$ is the identity operator and the potential operators

$$(\mathbf{S}_{S,\Gamma}\phi)(x) = 2\int_S G_1(x,y)\phi(y)\mathrm{d}s_y, \quad (\mathbf{K}_{S,\Gamma}\phi)(x) = 2\int_S \frac{\partial G_1(x,y)}{\partial \nu_S(y)}\phi(y)\mathrm{d}s_y, \quad x \in \Gamma,$$

and

$$(\mathbf{K}_{\Gamma,S}\phi)(x) = \int_\Gamma \frac{\partial G_1(x,y)}{\partial \nu_\Gamma(y)}\phi(y)\mathrm{d}s_y, \quad (\mathbf{T}_{\Gamma,S}\phi)(x) = \int_\Gamma \frac{\partial^2 G_1(x,y)}{\partial \nu_S(x)\partial \nu_\Gamma(y)}\phi(y)\mathrm{d}s_y, \quad x \in S,$$

have continuous kernels, which decay exponentially. It follows from [14, Theorem 1.10] that these operators are compact.

3.4 Solvability

Since the kernels of the operators $\mathbf{K}_\Gamma, \mathbf{K}_\Gamma^*, \mathbf{S}_A, \mathbf{K}_A, \mathbf{K}_A^*$ and $\mathbf{T}_A$ are weakly singular and decay exponentially, it follows from [14, Theorem 1.11] and [14, Theorem 2.7] that these integral operators are compact. Based on the compactness of the truncated operators, the integral operators $\mathbf{S}_S, \mathbf{K}_S, \mathbf{K}_S^*$ and $\mathbf{T}_S$ are compact as described in the following theorem.

Lemma 3.1. *The integral operators* $\mathbf{S}_S, \mathbf{K}_S, \mathbf{K}_S^*$ *and* $\mathbf{T}_S$ *are compact.*

Proof. Since the proofs are similar for $\mathbf{S}_S, \mathbf{K}_S, \mathbf{K}_S^*$ and $\mathbf{T}_S$, we shall only show the details for the operator $\mathbf{S}_S$. For each fixed $x \in S$, we have

$$(\mathbf{S}_S\phi_2)(x) - (\mathbf{S}_A\phi_2)(x) = \int_{S\setminus S_A} \big(G_1(x,y) - G_2(x,y)\big)\phi_2(y)\mathrm{d}s_y$$
$$= \left(\int_A^{+\infty} + \int_{-\infty}^{-A} \right) \Psi(x,y_1)\mathrm{d}y_1 := I_1 + I_2, \tag{3.32}$$

where

$$\Psi(x,y_1)=[(G_1(x,y)-G_2(x,y))\phi_2(y)|_{y_2=f(y_1)}](1+f_{y_1}^2)^{1/2}.$$

By Lemma 2.1, for each fixed $x\in S$, when $A\to+\infty$, we have $|x_1-A|\to+\infty$ and

$$\begin{aligned}
|I_1|&\le\int_A^{+\infty}|\Psi(x,y_1)|\mathrm{d}y_1\le C\int_A^{+\infty}|(G_1(x,y)-G_2(x,y))\phi_2(y)|_{y_2=f(y_1)}\mathrm{d}y_1\\
&\le C\sup_{y\in S}|\phi_2(y)|\int_A^{+\infty}|G_1(x,y)-G_2(x,y)|_{y_2=f(y_1)}\mathrm{d}y_1\\
&\le C\|\phi_2\|_{C^{0,\alpha}(S)}\int_A^{+\infty}\left(\frac{\exp(-\frac{1}{2}\Im(k_1)|y|)}{|y|^{\frac{1}{2}}}\Big|_{y_2=f(y_1)}+\frac{\exp(-\frac{1}{2}\Im(k_2)|y|)}{|y|^{\frac{1}{2}}}\Big|_{y_2=f(y_1)}\right)\mathrm{d}y_1\\
&\le C\|\phi_2\|_{C^{0,\alpha}(S)}\int_A^{+\infty}\left(\frac{\exp(-\frac{1}{2}\hat{k}|y_1|)}{|y_1|^{\frac{1}{2}}}\right)\mathrm{d}y_1\\
&\le C\|\phi_2\|_{C^{0,\alpha}(S)}\left(\frac{1}{\sqrt{A}}\right)\int_A^{+\infty}\exp\left(-\frac{1}{2}\hat{k}y_1\right)\mathrm{d}y_1\\
&=C\|\phi_2\|_{C^{0,\alpha}(S)}\left(\frac{2}{\hat{k}}\right)\left(\frac{1}{\sqrt{A}}\exp\left(-\frac{A}{2}\hat{k}\right)\right)\to 0,\quad A\to+\infty,
\end{aligned}\tag{3.33}$$

where $C>0$ is a constant but may change from step by step, $\hat{k}=\min\{\Im(k_1),\Im(k_2)\}>0$. Similarly, we may also show that

$$|I_2|\le C\|\phi_2\|_{C^{0,\alpha}(S)}\left(\frac{1}{\sqrt{A}}\exp\left(-\frac{A}{2}\hat{k}\right)\right)\to 0,\quad A\to+\infty.\tag{3.34}$$

Combining (3.32)-(3.34), we obtain

$$\begin{aligned}
|(\mathbf{S}_S\phi_2)(x)-(\mathbf{S}_A\phi_2)(x)|&\le|I_1|+|I_2|\\
&\le C\|\phi_2\|_{C^{0,\alpha}(S)}\left(\frac{1}{\sqrt{A}}\exp\left(-\frac{A}{2}\hat{k}\right)\right)\to 0,\quad A\to+\infty,
\end{aligned}$$

which implies that

$$\|\mathbf{S}_S-\mathbf{S}_A\|_\infty=\frac{\sup_{x\in S}|(\mathbf{S}_S\phi_2)(x)-(\mathbf{S}_A\phi_2)(x)|}{\|\phi_2\|_{C^{0,\alpha}(S)}}\le C\left(\frac{1}{\sqrt{A}}\exp\left(-\frac{A}{2}\hat{k}\right)\right)\to 0,\quad A\to+\infty.\tag{3.35}$$

Note that

$$\begin{aligned}
[\nabla_x(\mathbf{S}_S\phi_2)(x)]-[\nabla_x(\mathbf{S}_A\phi_2)(x)]&=\int_{S\backslash S_A}(\nabla_x G_1(x,y)-\nabla_x G_2(x,y))\phi_2(y)\mathrm{d}s_y\\
&=\left(\int_A^{+\infty}+\int_{-\infty}^{-A}\right)\nabla_x\Psi(x,y_1)\mathrm{d}y_1,
\end{aligned}$$

where

$$\nabla_x\Psi(x,y_1)=[(\nabla_x G_1(x,y)-\nabla_x G_2(x,y))\phi_2(y)|_{y_2=f(y_1)}](1+f_{y_1}^2)^{1/2}.$$

For each fixed $x\in S$, using Lemma 2.1 and repeating a proof similar to (3.33) and (3.34), we obtain

$$|[\nabla_x(\mathbf{S}_S\phi_2)(x)]-[\nabla_x(\mathbf{S}_A\phi_2)(x)]|\le C\|\phi_2\|_{C^{0,\alpha}(S)}\left(\frac{1}{\sqrt{A}}\exp\left(-\frac{A}{2}\hat{k}\right)\right)\to 0,\quad A\to+\infty,$$

which implies that

$$\begin{aligned}\|\nabla_x\mathbf{S}_S-\nabla_x\mathbf{S}_A\|_\infty&=\frac{\sup_{x\in S}|[\nabla_x(\mathbf{S}_S\phi_2)(x)]-[\nabla_x(\mathbf{S}_A\phi_2)(x)]|}{\|\phi_2\|_{C^{0,\alpha}(S)}}\\&\le C\left(\frac{1}{\sqrt{A}}\exp\left(-\frac{A}{2}\hat{k}\right)\right)\to 0,\quad A\to+\infty.\end{aligned}\tag{3.36}$$

For each fixed x, $\tilde{x}\in S$ and $x\neq\tilde{x}$, we have

$$\begin{aligned}&\nabla_x((\mathbf{S}_S-\mathbf{S}_A)\phi_2)(x)-\nabla_{\tilde{x}}((\mathbf{S}_S-\mathbf{S}_A)\phi_2)(\tilde{x})\\&=\left(\int_A^{+\infty}+\int_{-\infty}^{-A}\right)[\nabla_x\Psi(x,y_1)-\nabla_{\tilde{x}}\Psi(\tilde{x},y_1)]\mathrm{d}y_1:=I_3+I_4.\end{aligned}\tag{3.37}$$

From the mean value theorem and Lemma 2.1, when $A\to+\infty$, we have $|x_1-A|\to+\infty$, $|\tilde{x}_1-A|\to+\infty$ and

$$|\nabla_x G_j(x,y)-\nabla_{\tilde{x}}G_j(\tilde{x},y)|\le C\frac{\exp(-\frac{1}{2}\Im(k_j)|y|)}{|y|^{\frac{1}{2}}}|x-\tilde{x}|,\quad A\to+\infty,\quad j=1,2.$$

Hence,

$$\begin{aligned}|I_3|&\le\int_A^{+\infty}|\nabla_x\Psi(x,y_1)-\nabla_{\tilde{x}}\Psi(\tilde{x},y_1)|\mathrm{d}y_1\\&\le C\int_A^{+\infty}[(|\nabla_x G_1(x,y)-\nabla_{\tilde{x}}G_1(\tilde{x},y)|+|\nabla_x G_2(x,y)-\nabla_{\tilde{x}}G_2(\tilde{x},y)|)|\phi_2(y)|]_{y_2=f(y_1)}\mathrm{d}y_1\\&\le C(|x-\tilde{x}|)\sup_{y\in S}|\phi_2(y)|\int_A^{+\infty}\left(\frac{\exp(-\frac{1}{2}\Im(k_1)|y|)}{|y|^{\frac{1}{2}}}\bigg|_{y_2=f(y_1)}\right.\\&\qquad\left.+\frac{\exp(-\frac{1}{2}\Im(k_2)|y|)}{|y|^{\frac{1}{2}}}\bigg|_{y_2=f(y_1)}\right)\mathrm{d}y_1\\&\le C(|x-\tilde{x}|)\|\phi_2\|_{C^{0,\alpha}(S)}\left(\frac{2}{\hat{k}}\right)\left(\frac{1}{\sqrt{A}}\exp\left(-\frac{A}{2}\hat{k}\right)\right).\end{aligned}\tag{3.38}$$

Similarly, we have

$$|I_4| \leq C(|x-\tilde{x}|)\|\phi_2\|_{C^{0,\alpha}(S)}\left(\frac{1}{\sqrt{A}}\exp\left(-\frac{A}{2}\hat{k}\right)\right). \tag{3.39}$$

Using (3.37)-(3.39), we obtain for $0<\alpha<1$ that

$$\begin{aligned}
&\frac{|\nabla_x((\mathbf{S}_S-\mathbf{S}_A)\phi_2)(x)-\nabla_x((\mathbf{S}_S-\mathbf{S}_A)\phi_2)(\tilde{x})|}{|x-\tilde{x}|^\alpha}\\
\leq&(|I_3|+|I_4|)|x-\tilde{x}|^{-\alpha}\leq C(|x-\tilde{x}|^{1-\alpha})\|\phi_2\|_{C^{0,\alpha}(S)}\left(\frac{1}{\sqrt{A}}\exp\left(-\frac{A}{2}\hat{k}\right)\right)\\
\leq&CA^{1-\alpha}\|\phi_2\|_{C^{0,\alpha}(S)}\left(\frac{1}{\sqrt{A}}\exp\left(-\frac{A}{2}\hat{k}\right)\right)\\
=&C\|\phi_2\|_{C^{0,\alpha}(S)}\left(A^{\frac{1}{2}-\alpha}\exp\left(-\frac{A}{2}\hat{k}\right)\right)\to 0,\quad A\to+\infty.
\end{aligned} \tag{3.40}$$

For $0<\alpha<1$, by (3.35), (3.36) and (3.40), it can be deduced that

$$\begin{aligned}
&\|\mathbf{S}_S-\mathbf{S}_A\|_{C^{1,\alpha}(S)}\\
=&\sup_{\|\phi_2\|_{C^{0,\alpha}(S)}\neq 0}\frac{\|(\mathbf{S}_S-\mathbf{S}_A)\phi_2\|_{C^{1,\alpha}(S)}}{\|\phi_2\|_{C^{0,\alpha}(S)}}\\
=&\sup_{\|\phi_2\|_{C^{0,\alpha}(S)}\neq 0}\frac{1}{\|\phi_2\|_{C^{0,\alpha}(S)}}\Bigg[\|(\mathbf{S}_S-\mathbf{S}_A)\phi_2\|_\infty+\|\nabla_x(\mathbf{S}_S-\mathbf{S}_A)\phi_2\|_\infty\\
&+\sup_{\substack{x,\tilde{x}\in S\\ x\neq\tilde{x}}}\frac{|\nabla_x((\mathbf{S}_S-\mathbf{S}_A)\phi_2)(x)-\nabla_x((\mathbf{S}_S-\mathbf{S}_A)\phi_2)(\tilde{x})|}{|x-\tilde{x}|^\alpha}\Bigg]\\
\leq&C\left(\sqrt{A}\exp\left(-\frac{A}{2}\hat{k}\right)\right)\to 0,\quad A\to+\infty,
\end{aligned}$$

which shows that the operator $\mathbf{S}_S$ is compact on $C^{0,\alpha}(S)$. □

By Lemma 3.1, the system (3.31) is Fredholm, which implies that the existence of a solution to (3.31) follows from the uniqueness of the solution. Although the direct scattering problem is shown to have a unique solution in Theorem 3.1, the boundary integral equations (3.31) may not have a unique solution due to the possible existence of resonance. This issue can be overcome by using the combined single- and double-layer potentials. We will not elaborate on this issue and leave it along with the numerical solution of (3.31) for a future work.

4 Uniqueness of the inverse scattering problem

This section addresses the global uniqueness of the inverse scattering problem. Given the incident field which satisfies (2.3), we show that the obstacle and the infinite rough surface can be uniquely determined by the wave field $u_j|_{\Gamma_j}$, $j=1,2$.

Let $\widetilde{S}\in C^2$ be an infinite rough surface which divides $\mathbb{R}^n$ into the upper half space $\widetilde{\Omega}_1^+$ and the lower half space $\widetilde{\Omega}_2$. Let $\widetilde{D}\subset\subset\widetilde{\Omega}_1^+$ be a bounded obstacle with the boundary $\widetilde{\Gamma}\in C^2$. Define $\widetilde{\Omega}_1=\widetilde{\Omega}_1^+\setminus\overline{\widetilde{D}}$. Let $(\tilde{u}_1,\tilde{u}_2)$ be the unique solutions of Problem 2.1 with D replaced by $\widetilde{D}$ and S replaced by $\widetilde{S}$ but for the same incident field u^i satisfying (2.3), where $\tilde{u}_1$ satisfies the sound hard boundary condition

$$\frac{\partial\tilde{u}_1}{\partial\nu_{\widetilde{\Gamma}}}=0\quad\text{on }\widetilde{\Gamma},$$

and the continuity conditions

$$\tilde{u}_1=\tilde{u}_2,\quad\frac{\partial\tilde{u}_1}{\partial\nu_{\widetilde{S}}}=\frac{\partial\tilde{u}_2}{\partial\nu_{\widetilde{S}}}\quad\text{on }\widetilde{S},$$

where $\nu_{\widetilde{\Gamma}}$ be the unit normal vector on $\widetilde{\Gamma}$ directed into the exterior of $\widetilde{D}$ and let $\nu_{\widetilde{S}}$ be the unit normal vector on the boundary $\widetilde{S}$ pointing from region $\widetilde{\Omega}_2$ to region $\widetilde{\Omega}_1$. Since the point source is assumed to be located in R_1^+, we have $x_s\in\Omega_1\cap\widetilde{\Omega}_1$. In addition, $\tilde{u}_1$ satisfies the transparent boundary condition (2.9).

Lemma 4.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded Lipschitz domain. Consider the boundary value problem*

$$\begin{cases}\Delta u+k_1^2u=0,\quad\Delta v+k_2^2v=0 & \text{in }\Omega,\\ u=v,\quad\frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu} & \text{on }\partial\Omega,\end{cases}\tag{4.1}$$

where ν is the unit normal vector on Ω. Then $u=v=0$ in Ω.

Proof. Consider an extension $\breve{v}$ of v to the exterior domain $\Omega^e=\mathbb{R}^2\setminus\overline{\Omega}$, where $\breve{v}$ satisfies

$$\begin{cases}\Delta\breve{v}+k_2^2\breve{v}=0 & \text{in }\Omega^e,\\ \breve{v}=v,\quad\frac{\partial\breve{v}}{\partial\nu}=\frac{\partial v}{\partial\nu} & \text{on }\partial\Omega\end{cases}$$

and the radiation condition

$$\lim_{r\to\infty}\int_{\partial B_r}\left(|\breve{v}|^2+\left|\frac{\partial\breve{v}}{\partial\nu}\right|^2\right)\mathrm{d}s=0.$$

Multiplying the equation $\Delta u+k_1^2u=0$ by the complex conjugate of u, integrating over Ω, and using the integration by parts, we have

$$\int_\Omega|\nabla u|^2\mathrm{d}x-k_1^2\int_\Omega|u|^2\mathrm{d}x=\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\overline{u}\mathrm{d}s.$$

On the other hand, multiplying the equation $\Delta\breve{v}+k_2^2\breve{v}=0$ by the complex conjugate of $\breve{v}$, integrating over Ω^e, using the integration by parts and the radiation condition, we obtain

$$\int_{\Omega^e}|\nabla\breve{v}|^2\mathrm{d}x-k_2^2\int_{\Omega^e}|\breve{v}|^2\mathrm{d}x=-\int_{\partial\Omega}\frac{\partial\breve{v}}{\partial\nu}\bar{\breve{v}}\mathrm{d}s.$$

Since $u=v=\breve{v}$ and $\frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=\frac{\partial\breve{v}}{\partial\nu}$ on $\partial\Omega$, we add the above two equation and get

$$\int_{\Omega}|\nabla u|^2\mathrm{d}x+\int_{\Omega^e}|\nabla\breve{v}|^2\mathrm{d}x-k_1^2\int_{\Omega}|u|^2\mathrm{d}x-k_2^2\int_{\Omega^e}|\breve{v}|^2\mathrm{d}x=0.$$

Noting $\Im k_j^2>0$ and taking the imaging part of the above equation yields that $u=0$ in Ω and $\breve{v}=0$ in Ω^e, which implies immediately that $u=v=0$ in Ω. □

Remark 4.1. The result still holds for $k_1=k_2$ in Lemma 4.1. In this case, the problem (4.1) is equivalent to the following scattering problem: To find u such that it satisfies the Helmholtz equation

$$\Delta u+k_1^2u=0\quad\text{in }\mathbb{R}^2$$

and the radiation condition

$$\lim_{r\to\infty}\int_{\partial B_r}\left(|u|^2+\left|\frac{\partial u}{\partial\nu}\right|^2\right)\mathrm{d}s=0.$$

It is clear to note that the above scattering problem has a unique solution $u=0$ in $\mathbb{R}^2$ due to $\Im k_1^2>0$.

Lemma 4.2. *Let u_1 be the solution of Problem* 2.1. *Then $u_1\neq0$ on Γ or S.*

Proof. We prove it by contradiction. First we assume that $u_1=0$ on Γ. Recall the sound hard boundary condition $\frac{\partial u_1}{\partial\nu_\Gamma}=0$ on Γ. Consider an extension $\breve{u}_1$ of u_1 to the domain Ω_1^+:

$$\breve{u}_1:=\begin{cases}u_1 & \text{in }\Omega_1,\\ v_1 & \text{in }D,\end{cases}$$

where v_1 satisfies

$$\begin{cases}\Delta v_1+k_1^2v_1=0 & \text{in }D,\\ v_1=u_1,\quad \frac{\partial v_1}{\partial\nu_\Gamma}=\frac{\partial u_1}{\partial\nu_\Gamma} & \text{on }\Gamma.\end{cases}\tag{4.2}$$

Hence the extension $\breve{u}_1$ satisfies

$$\begin{cases}\Delta\breve{u}_1+k_1^2\breve{u}_1=0 & \text{in }R_1,\\ \frac{\partial\breve{u}_1}{\partial\nu_{\Gamma_1}}=T_1\breve{u}_1+\rho_1 & \text{on }\Gamma_1.\end{cases}\tag{4.3}$$

Since $\breve{u}_1=u_1=0$ on Γ and $\Im k_1^2>0$, it is easy to verify from (4.2) that $\breve{u}_1=v_1=0$ in D. It follows from the unique continuation that $\breve{u}_1=0$ in R_1, which contradicts the transparent boundary condition in (4.3).

Next we assume that $u_1=0$ on S. Since $u_2=u_1=0$ on S, we may consider the following problem

$$\begin{cases} \Delta u_2+k_2^2u_2=0 & \text{in } \Omega_2, \\ u_2=0 & \text{on } S. \end{cases} \tag{4.4}$$

In addition, u_2 is required to satisfy the radiation condition (2.10). Multiplying the complex conjugate of u_2, integrating over Ω_2, and using the radiation condition, we obtain

$$\int_{\Omega_2}|\nabla u_2|^2\mathrm{d}x-k_2^2\int_{\Omega_2}|u_2|^2\mathrm{d}x=0,$$

which implies $u_2=0$ in Ω_2 due to $\Im k_2^2>0$. Hence $\frac{\partial u_2}{\partial \nu_S}=0$ on S. Since $u_1=u_2=0$ and $\frac{\partial u_1}{\partial \nu_S}=\frac{\partial u_2}{\partial \nu_S}=0$ on S, by the Holmgren uniqueness theorem, $u_1=0$ in R_1. In fact, u_1 can be extended to $\Omega_1\cup\Omega_2$ as follows

$$\breve{u}_1:=\begin{cases} u_1 & \text{in } \Omega_1, \\ v_1 & \text{in } \Omega_2, \end{cases}$$

where v_1 satisfies

$$\begin{cases} \Delta v_1+k_1^2v_1=0 & \text{in } \Omega_2, \\ v_1=u_1, \quad \frac{\partial v_1}{\partial \nu_S}=\frac{\partial u_1}{\partial \nu_S} & \text{on } S \end{cases} \tag{4.5}$$

and the radiation condition (2.10). Clearly the problem (4.5) has a unique solution $v_1=0$ in Ω_2. By the unique continuation, we have $u_1=\breve{u}_1=0$ in R_1, which contradicts the transparent boundary condition (2.9). □

Theorem 4.1. *Assume that $u_j|_{\Gamma_j}=\tilde{u}_j|_{\Gamma_j}$, $j=1,2$ for the given the incident wave u^i. Then $D=\widetilde{D}$ and $S=\widetilde{S}$.*

Proof. Let $v_j=u_j-\tilde{u}_j$, then v_j satisfies the Helmholtz equation

$$\Delta v_j+k_j^2v_j=0 \quad \text{in } \Omega_j\cap\widetilde{\Omega}_j$$

and the radiation condition. By the assumption $v_j|_{\Gamma_j}=u_j|_{\Gamma_j}-\tilde{u}_j|_{\Gamma_j}$ and the uniqueness result for the scattering problem, one can obtain that $v_j=u_j-\tilde{u}_j=0$ and $\nabla v_j=\nabla u_j-\nabla\tilde{u}_j=0$, $j=1,2$ in $R_1^+=\{x\in\mathbb{R}^2:x_n>h_1\}$ and $R_2^+=\{x\in\mathbb{R}^2:x_n<h_2\}$, respectively. Since $v_j\in C^2(\Omega_j\cap\widetilde{\Omega}_j)\cap C^{1,\alpha}(\overline{\Omega_j\cap\widetilde{\Omega}_j})$, by the unique continuation, we get that

$$v_j(x)=u_j(x)-\tilde{u}_j(x)=0, \quad x\in\overline{\Omega_j\cap\widetilde{\Omega}_j} \tag{4.6}$$

and

$$\frac{\partial v_j(x)}{\partial \nu}=\frac{\partial u_j(x)}{\partial \nu}-\frac{\partial \tilde{u}_j(x)}{\partial \nu}=0, \quad x\in\partial(\overline{\Omega_j\cap\widetilde{\Omega}_j}). \tag{4.7}$$

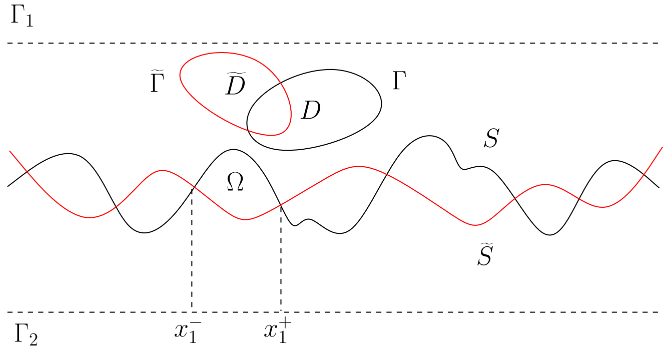

Figure 2: Schematic of domains for the proof of uniqueness.

First, we show the obstacle can be uniquely determined by the total field u_1 on Γ_1. Assume that $D \neq \widetilde{D}$. Then $D\backslash(\overline{D\cap\widetilde{D}}) \neq \emptyset$ or $\widetilde{D}\backslash(\overline{D\cap\widetilde{D}}) \neq \emptyset$. The schematic of the domains D and $\widetilde{D}$ is shown in Fig. 2. Without loss of generality, we assume that $\widetilde{D}\backslash(\overline{D\cap\widetilde{D}}) \neq \emptyset$. Denote $\partial(\widetilde{D}\backslash(\overline{D\cap\widetilde{D}}))$ by $\widetilde{C}\cup C$, where $\widetilde{C}\subset\widetilde{\Gamma}$ and $C\subset\Gamma$. Since $\frac{\partial\tilde{u}_1}{\partial\nu_{\widetilde{\Gamma}}}=0$ on $\widetilde{\Gamma}$, we have from (4.6)-(4.7) that $\frac{\partial u_1}{\partial\nu_{\widetilde{\Gamma}}}=0$ on $\widetilde{C}$. Recalling $\frac{\partial u_1}{\partial\nu_\Gamma}=0$ on C, we consider the following boundary value problem

$$\begin{cases} \Delta u_1 + k_1^2 u_1 = 0 & \text{in } \widetilde{D}\backslash(\overline{D\cap\widetilde{D}}), \\ \frac{\partial u_1}{\partial\nu_{\widetilde{\Gamma}}} = 0 & \text{on } \widetilde{C}, \\ \frac{\partial u_1}{\partial\nu_\Gamma} = 0 & \text{on } C. \end{cases}$$

Multiplying $\Delta u_1 + k_1^2 u_1 = 0$ by the complex conjugate of u_1, integrating over $\widetilde{D}\backslash(\overline{D\cap\widetilde{D}})$, using the integration by parts and the boundary conditions, we obtain

$$\int_{\widetilde{D}\backslash(\overline{D\cap\widetilde{D}})} |\nabla u_1|^2 \mathrm{d}x - k_1^2 \int_{\widetilde{D}\backslash(\overline{D\cap\widetilde{D}})} |u_1|^2 \mathrm{d}x = 0,$$

which implies that $u_1 = 0$ in $\widetilde{D}\backslash(\overline{D\cap\widetilde{D}})$ since $\Im k_1^2 > 0$. An application of the unique continuation gives $u_1 = 0$ in R_1. But this contradicts the transparent boundary condition (2.9) on Γ_1 since ρ is a nonzero function involving the incident wave. Hence $D = \widetilde{D}$.

Next is show that the infinite rough surface S can be uniquely determined by the wave fields u_1 and u_2 measured on Γ_1 and Γ_2, respectively. Assume that $S \neq \widetilde{S}$, where $\widetilde{S} = \{x \in \mathbb{R}^2 : x_2 = \tilde{f}(x_1)\}$ with $\tilde{f} \neq f$. Without loss of generality, we assume that there exist two points x_1^-, x_1^+ satisfying $x_1^- < x_1^+$ such that $f(x_1^-) = \tilde{f}(x_1^-)$, $f(x_1^+) = \tilde{f}(x_1^+)$ and $f(x_1) > \tilde{f}(x_1)$ for $x_1 \in (x_1^-, x_1^+)$. Define $\Omega = \{x \in \mathbb{R}^2 : \tilde{f}(x_1) < x_2 < f(x_1), x_1^- < x_1 < x_1^+\}$. The schematic of the domain Ω is also shown in Fig. 2. Let $\partial\Omega = \Sigma\cup\widetilde{\Sigma}$, where $\Sigma\subset S$ and $\widetilde{\Sigma}\subset\widetilde{S}$. By (4.6)-(4.7), we have

$$u_1 = \tilde{u}_1, \quad \frac{\partial u_1}{\partial\nu} = \frac{\partial\tilde{u}_1}{\partial\nu} \quad \text{on } \Sigma$$

and

$$u_2=\tilde{u}_2,\quad \frac{\partial u_2}{\partial \nu}=\frac{\partial \tilde{u}_2}{\partial \nu}\quad \text{on } \widetilde{\Sigma}.$$

It follows from the continuity conditions

$$u_1=u_2,\quad \frac{\partial u_1}{\partial \nu}=\frac{\partial u_2}{\partial \nu}\quad \text{on } \Sigma$$

and

$$\tilde{u}_1=\tilde{u}_2,\quad \frac{\partial \tilde{u}_1}{\partial \nu}=\frac{\partial \tilde{u}_2}{\partial \nu}\quad \text{on } \widetilde{\Sigma}.$$

Combining the above equations yields that

$$\tilde{u}_1=u_2,\quad \frac{\partial \tilde{u}_1}{\partial \nu}=\frac{\partial u_2}{\partial \nu}\quad \text{on } \Sigma\cup\widetilde{\Sigma}.$$

We consider the following boundary value problem

$$\begin{cases} \Delta\tilde{u}_1+k_1^2\tilde{u}_1=0, \quad \Delta u_2+k_2^2u_2=0 & \text{in } \Omega, \\ \tilde{u}_1=u_2, \quad \frac{\partial \tilde{u}_1}{\partial \nu}=\frac{\partial u_2}{\partial \nu} & \text{on } \partial\Omega. \end{cases}$$

It follows from Lemma 4.1 that $\tilde{u}_1=u_2=0$ in Ω. An application of the unique continuation gives $\tilde{u}_1=0$ in R_1, which contradicts the transparent boundary condition (2.9). So we have $S=\widetilde{S}$. □

5 Local stability

In applications, it is impossible to make exact measurements. Stability is crucial in the practical reconstruction since it contains necessary information to determine to what extent the data can be trusted.

Let $I:\mathbb{R}^2\to\mathbb{R}^2$ be the identity mapping or unit matrix and let $\theta:\Gamma\cup S\to\mathbb{R}^2$ be an admissible perturbation, where θ is assumed to be an admissible perturbation in $C^2(\Gamma\cup S,\mathbb{R}^2)$ and has a compact support. For $\theta\in C^2(\Gamma\cup S,\mathbb{R}^2)$, we can extend the definition of function $\theta(x)$ to $\overline{\Omega}_j$ by requiring that $\theta(x)\in C^2(\Omega_j,\mathbb{R}^2)\cap C(\overline{\Omega}_j,\mathbb{R}^2)$; $I+\theta:\Omega_j\to\Omega_{j,\theta}$, $j=1,2$. Here the region $\Omega_{j,\theta}$ bounded by Γ_θ and S_θ, where

$$\Gamma_\theta=\{x+\theta(x):x\in\Gamma\},\quad S_\theta=\{x+\theta(x):x\in S\}.$$

Here $\theta(x)=(\theta_1(x),\theta_2(x))^\top$. According to Theorem 3.1, there exist the unique solutions $(u_{1,\theta},u_{2,\theta})$ to Problem 2.1 corresponding to the region $\Omega_{j,\theta}$ for any small enough θ. The map $\theta\to u_\theta$ is locally differentiable if for every open set D strictly included in Ω_j and strictly included in $\Omega_{j,\theta}$, the restriction of u_θ to D is differentiable [39]. The domain derivative of $u_{j,\theta}$ at $\theta=0$ in the direction p is given by

$$u_j'=\frac{\partial u_{j,\theta}}{\partial \theta}(0)p,\quad j=1,2,$$

where $\frac{\partial u_{j,\theta}}{\partial\theta}(0)=(\frac{\partial u_{j,\theta}}{\partial\theta_1},\frac{\partial u_{j,\theta}}{\partial\theta_2})|_{\theta=0}$ and $p(x)=(p_1(x),p_2(x))^\top\in C^2(\Gamma\cup S,\mathbb{R}^2)$. Define a nonlinear map

$$F:\Gamma_\theta\cup S_\theta\to u_{1,\theta}|_{\Gamma_1}.$$

The domain derivative of the operator F on the boundary $\Gamma\cup S$ along the direction p is defined by

$$F'(\Gamma\cup S,p):=u_1'|_{\Gamma_1}.$$

Define the jumps on S:

$$[u_1(x)-u_2(x)]=\lim_{\substack{a_1\to 0\\ x+a_1\in\Omega_1}}u_1(x+a_1)-\lim_{\substack{a_2\to 0\\ x+a_2\in\Omega_2}}u_2(x+a_2)$$

and

$$[\nu_S\cdot\nabla u_1(x)-\nu_S\cdot\nabla u_2(x)]=\lim_{h\to+0}\nu_S\cdot\nabla u_1(x+h\nu_S)-\lim_{h\to+0}\nu_S\cdot\nabla u_2(x-h\nu_S).$$

Theorem 5.1. *Let (u_1,u_2) be the solutions of Problem* 2.1*. Given $p\in C^2(\Gamma\cup S,\mathbb{R}^2)$, the domain derivatives (u_1',u_2') of (u_1,u_2) are the radiating solutions of the following problem:*

$$\begin{cases}\Delta u_1'+k_1^2u_1'=0 & \text{in } \Omega_1,\\ \Delta u_2'+k_2^2u_2'=0 & \text{in } \Omega_2,\\ \nu_\Gamma\cdot(\nabla u_1')=k_1^2p_{\Gamma_\nu}u_1+\nabla\cdot[p_{\Gamma_\nu}(\nabla_{\Gamma_\tau}u_1)] & \text{on } \Gamma,\\ [u_1'-u_2']=0 & \text{on } S,\\ [(\nu_S\cdot\nabla u_1')-(\nu_S\cdot\nabla u_2')]=p_{S_\nu}[k_1^2u_1-k_2^2u_2] & \text{on } S.\end{cases}\tag{5.1}$$

Proof. Define the operator

$$\mathcal{A}=\Delta+k_1^2I.$$

Let

$$\omega_\theta=\mathcal{A}u_{1,\theta},\tag{5.2}$$

then, we have

$$\omega_\theta=-\delta\quad\text{in }\Omega_{1,\theta}\tag{5.3}$$

and

$$\omega_\theta(I+\theta)=-\delta\quad\text{in }\Omega_1.\tag{5.4}$$

Since $\mathcal{A}$ is a linear and continuous operator from $H^1_{\rm loc}(\Omega_1)$ into $\mathcal{D}'(\Omega_1)$, $\mathcal{A}$ is differentiable in the distribution sense, i.e., $\boldsymbol{v}\mapsto\langle\mathcal{A}\boldsymbol{v},\boldsymbol{\psi}\rangle$ is differentiable for each $\boldsymbol{\psi}\in\mathcal{D}(\Omega_1)$ and

$$\frac{\partial\mathcal{A}}{\partial\boldsymbol{v}}=\mathcal{A}.\tag{5.5}$$

Here $\mathcal{D}(\Omega_1)$ is the space of infinitely differentiable functions with compact support in Ω_1 and $\mathcal{D}'(\Omega_1)$ is the space of distributions. Therefore, it follows from the differentiability of $\theta\mapsto u_{1,\theta}(I+\theta)$ and $\theta\mapsto u_{1,\theta}$ that $\theta\mapsto \omega_\theta(I+\theta)$ is Fréchet differentiable at $\theta=0$ in the direction $p\in C^2(\Gamma\cup S,\mathbb{R}^2)$. Moreover, for an admissible perturbation θ, i.e., $\theta\in C^2(\Gamma\cup S,\mathbb{R}^2)$, and θ in the neighborhood of 0, from (5.3) and (5.4), their derivatives satisfy

$$\frac{\partial\omega_\theta}{\partial\theta}(0)p=\frac{\partial}{\partial\theta}(\omega_\theta(I+\theta))(0)p-(p\cdot\nabla)\omega=-(p\cdot\nabla)\delta+(p\cdot\nabla)\delta=0 \quad \text{in } \Omega_1. \tag{5.6}$$

We deduce from (5.2) and (5.6) that

$$\frac{\partial\omega_\theta}{\partial\theta}(0)p=\left(\frac{\partial\mathcal{A}}{\partial u_{1,\theta}}\frac{\partial u_{1,\theta}}{\partial\theta}\right)(0)p=\frac{\partial\mathcal{A}}{\partial u_1}u_1'=0 \quad \text{in } \Omega_1. \tag{5.7}$$

It follows from (5.5) and (5.7) that

$$\mathcal{A}u_1'=\Delta u_1'+k_1^2u_1'=0 \quad \text{in } \Omega_1.$$

Furthermore, for every perturbation $\theta\in C^2(\Gamma\cup S,\mathbb{R}^2)$, the total fields and their normal derivative are assumed to satisfy

$$u_{1,\theta}=u_{2,\theta}, \quad \nu_{S_\theta}\cdot\nabla u_{1,\theta}=\nu_{S_\theta}\cdot\nabla u_{2,\theta} \quad \text{on } S_\theta.$$

Hence,

$$u_{1,\theta}(I+\theta)=u_{2,\theta}(I+\theta) \quad \text{on } S \tag{5.8}$$

and

$$[\nu_{S_\theta}(I+\theta)]\cdot[(\nabla u_{1,\theta})(I+\theta)]=[\nu_{S_\theta}(I+\theta)]\cdot[(\nabla u_{2,\theta})(I+\theta)] \quad \text{on } S. \tag{5.9}$$

We have from straightforward calculations that

$$\begin{aligned}&\frac{\partial}{\partial\theta}(u_{j,\theta}(I+\theta))(0)p=\frac{\partial u_{j,\theta}}{\partial\theta}(0)p+p\cdot(\nabla u_j)=u_j'+p\cdot(\nabla u_j)\\ &=u_j'+[p_{S_\tau}+p_{S_\nu}\nu_S]\cdot[\nabla_{S_\tau}u_j+(\nu_S\cdot\nabla u_j)\nu_S]\\ &=u_j'+[p_{S_\tau}\cdot(\nabla_{S_\tau}u_j)]+[p_{S_\nu}(\nu_S\cdot\nabla u_j)] \quad \text{on } S.\end{aligned} \tag{5.10}$$

Considering the boundary conditions (2.8) and $p\in C^2(\Gamma\cup S,\mathbb{R}^2)$, we have from (5.8) and (5.10) that

$$[u_1'-u_2']=-p_{S_\tau}\cdot[\nabla_{S_\tau}u_1-\nabla_{S_\tau}u_2]-p_{S_\nu}[(\nu_S\cdot\nabla u_1)-(\nu_S\cdot\nabla u_2)] \quad \text{on } S. \tag{5.11}$$

It follows from $[u_1-u_2]=0$ on S, and the definition of the surface gradient that

$$[\nabla_{S_\tau}u_1-\nabla_{S_\tau}u_2]=0, \quad \nabla(u_1-u_2)=[\nu_S\cdot\nabla(u_1-u_2)]\nu_S \quad \text{on } S. \tag{5.12}$$

From (5.11) and (5.12), the jump relations now yield

$$[u_1'-u_2']=-p_{S_\nu}[(\nu_S\cdot\nabla u_1)-(\nu_S\cdot\nabla u_2)]=0 \quad \text{on } S. \tag{5.13}$$

It follows from [15, Lemma 3] or [36, Lemma 4.8] that

$$\nu_{S_\theta}(I+\theta)=\frac{1}{\|g(\theta)\nu_S\|_2}g(\theta)\nu_S \quad \text{on } S \tag{5.14}$$

and

$$\nu_{\Gamma_\theta}(I+\theta)=\frac{1}{\|g(\theta)\nu_\Gamma\|_2}g(\theta)\nu_\Gamma \quad \text{on } \Gamma, \tag{5.15}$$

where the matrix $g(\theta)$ is given by

$$g(\theta)=[(I+\theta)']^{-\top}=\begin{bmatrix} 1+\frac{\partial\theta_1(x)}{\partial x_1} & \frac{\partial\theta_1(x)}{\partial x_2} \\ \frac{\partial\theta_2(x)}{\partial x_1} & 1+\frac{\partial\theta_2(x)}{\partial x_2}\end{bmatrix}^{-\top} \quad \text{on } \Gamma\cup S$$

and satisfies

$$g(0)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \frac{\partial g(\theta)}{\partial\theta}(0)p=-(\nabla p)^\top=-\begin{bmatrix}\frac{\partial p_1}{\partial x_1} & \frac{\partial p_1}{\partial x_2}\\ \frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial x_2}\end{bmatrix}^\top.$$

Therefore we get from (5.9) and (5.14) that

$$[g(\theta)\nu_S]\cdot[(\nabla u_{1,\theta})(I+\theta)]=[g(\theta)\nu_S]\cdot[(\nabla u_{2,\theta})(I+\theta)] \quad \text{on } S.$$

which implies that

$$[g(\theta)\nu_S]\cdot[g(\theta)\nabla(u_{1,\theta}(I+\theta))]=[g(\theta)\nu_S]\cdot[g(\theta)\nabla(u_{2,\theta}(I+\theta))] \quad \text{on } S. \tag{5.16}$$

Using the chain rule, we deduce from (5.16) that

$$\begin{aligned}&\frac{\partial}{\partial\theta}\{[g(\theta)\nu_S]\cdot[g(\theta)\nabla(u_{j,\theta}(I+\theta))]\}(0)p\\ =&\left[\left(\frac{\partial g(\theta)}{\partial\theta}(0)p\right)\nu_S\right]\cdot[g(0)(\nabla u_j)]+(g(0)\nu_S)\cdot\left[\left(\frac{\partial g(\theta)}{\partial\theta}(0)p\right)(\nabla u_j)\right]\\ &+(g(0)\nu_S)\cdot\left[g(0)\frac{\partial}{\partial\theta}(\nabla(u_{j,\theta}(I+\theta)))(0)p\right]\end{aligned}$$

$$
\begin{aligned}
&=\left[\left(\frac{\partial g(\theta)}{\partial\theta}(0)p\right)\nu_S\right]\cdot(\nabla u_j)+\nu_S\cdot\left[\left(\frac{\partial g(\theta)}{\partial\theta}(0)p\right)(\nabla u_j)\right]\\
&\quad+\nu_S\cdot\left[\nabla\left(\frac{\partial}{\partial\theta}(u_{j,\theta}(I+\theta))(0)p\right)\right]\\
&=-[(\nabla p)^\top\nu_S]\cdot(\nabla u_j)-\nu_S\cdot[(\nabla p)^\top(\nabla u_j)]+\nu_S\cdot[\nabla u_j'+\nabla(p\cdot(\nabla u_j))]\\
&=-[(\nabla p)^\top\nu_S]\cdot(\nabla u_j)-\nu_S\cdot[(\nabla p)^\top(\nabla u_j)]+\nu_S\cdot(\nabla u_j')+\nu_S\cdot[\nabla(p\cdot(\nabla u_j))]\\
&=-[(\nabla p)^\top\nu_S]\cdot(\nabla u_j)-\nu_S\cdot[(\nabla p)^\top(\nabla u_j)]+\nu_S\cdot(\nabla u_j')\\
&\quad+\nu_S\cdot\{p\times[\nabla\times(\nabla u_j)]+(p\cdot\nabla)(\nabla u_j)+[(\nabla u_j)\times(\nabla\times p)+((\nabla u_j)\cdot\nabla)p]\}\\
&=-[(\nabla p)^\top\nu_S]\cdot(\nabla u_j)-\nu_S\cdot[(\nabla p)^\top(\nabla u_j)]+\nu_S\cdot(\nabla u_j')\\
&\quad+\nu_S\cdot\{0+(p\cdot\nabla)(\nabla u_j)+[(\nabla u_j)\times(\nabla\times p)+((\nabla u_j)\cdot\nabla)p]\}\\
&=-[(\nabla p)^\top\nu_S]\cdot(\nabla u_j)-\nu_S\cdot[(\nabla p)^\top(\nabla u_j)]+\nu_S\cdot(\nabla u_j')\\
&\quad+\nu_S\cdot\{(p\cdot\nabla)(\nabla u_j)+[(\nabla u_j)\times(\nabla\times p)+((\nabla u_j)\cdot\nabla)p]\}\quad\text{on } S.
\end{aligned}\tag{5.17}
$$

Since on $S\cup\Gamma$, for $j=1,2$, one can easily verify that

$$
\begin{aligned}
&[(\nabla p)^\top(\nabla u_j)]=[(\nabla u_j)\times(\nabla\times p)+((\nabla u_j)\cdot\nabla)p],\\
&[(\nabla p)^\top\nu]=[\nu\times(\nabla\times p)+(\nu\cdot\nabla)p]=\nabla(p\cdot\nu)-[(p\cdot\nabla)\nu]-[p\times(\nabla\times\nu)]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad=\nabla(p\cdot\nu)-[(p\cdot\nabla)\nu],\\
&[(p\cdot\nabla)\nu]\cdot(\nabla u_j)+\nu\cdot[(p\cdot\nabla)(\nabla u_j)]=(p\cdot\nabla)(\nu\cdot\nabla u_j),\\
&[(p\cdot\nabla)\nu]\cdot\nu=\frac{1}{2}(p\cdot\nabla)(\nu\cdot\nu)=\frac{1}{2}(p\cdot\nabla)(|\nu|^2)=\frac{1}{2}(p\cdot\nabla)(1)=0.
\end{aligned}
$$

With the aid of (5.17), we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial\theta}\{[g(\theta)\nu_S]\cdot[g(\theta)\nabla(u_{j,\theta}(I+\theta))]\}(0)p\\
&=-[(\nabla p)^\top\nu_S]\cdot(\nabla u_j)-\nu_S\cdot[(\nabla p)^\top(\nabla u_j)]\\
&\qquad+\nu_S\cdot(\nabla u_j')+\nu_S\cdot[(\nabla p)^\top(\nabla u_j)]+\nu_S\cdot[(p\cdot\nabla)(\nabla u_j)]\\
&=-[(\nabla p)^\top\nu_S]\cdot(\nabla u_j)+\nu_S\cdot(\nabla u_j')+\nu_S\cdot[(p\cdot\nabla)(\nabla u_j)]\\
&=-[\nabla(p\cdot\nu_S)]\cdot(\nabla u_j)+[(p\cdot\nabla)\nu_S]\cdot(\nabla u_j)+\nu_S\cdot(\nabla u_j')+\nu_S\cdot[(p\cdot\nabla)(\nabla u_j)]\\
&=-[\nabla(p\cdot\nu_S)]\cdot(\nabla u_j)+\nu_S\cdot(\nabla u_j')+\{[(p\cdot\nabla)\nu_S]\cdot(\nabla u_j)+\nu_S\cdot[(p\cdot\nabla)(\nabla u_j)]\}\\
&=-[\nabla(p\cdot\nu_S)]\cdot(\nabla u_j)+\nu_S\cdot(\nabla u_j')+[(p\cdot\nabla)(\nu_S\cdot\nabla u_j)]\\
&=-\nabla\cdot[(p\cdot\nu_S)(\nabla u_j)]+(p\cdot\nu_S)[\nabla\cdot(\nabla u_j)]+\nu_S\cdot(\nabla u_j')+[(p\cdot\nabla)(\nu_S\cdot\nabla u_j)]\\
&=-\nabla\cdot[(p\cdot\nu_S)(\nabla u_j)]+(p\cdot\nu_S)(\Delta u_j)+\nu_S\cdot(\nabla u_j')+[(p\cdot\nabla)(\nu_S\cdot\nabla u_j)]\\
&=-\nabla\cdot[(p\cdot\nu_S)(\nabla u_j)]-(p\cdot\nu_S)(k_j^2u_j)+\nu_S\cdot(\nabla u_j')+[(p\cdot\nabla)(\nu_S\cdot\nabla u_j)]
\end{aligned}
$$

$$
\begin{aligned}
&=-\nabla\cdot[p_{S_\nu}(\nabla_{S_\tau}u_j)+p_{S_\nu}(\nu_S\cdot\nabla u_j)\nu_S]-p_{S_\nu}(k_j^2u_j)+\nu_S\cdot(\nabla u_j')+[(p\cdot\nabla)(\nu_S\cdot\nabla u_j)]\\
&=-\nabla\cdot[p_{S_\nu}(\nabla_{S_\tau}u_j)]-\nabla\cdot[(\nu_S\cdot\nabla u_j)(p_{S_\nu}\nu_S)]\\
&\quad -p_{S_\nu}(k_j^2u_j)+\nu_S\cdot(\nabla u_j')+[(p\cdot\nabla)(\nu_S\cdot\nabla u_j)]. \qquad (5.18)
\end{aligned}
$$

By the continuous conditions (2.8) and $p\in C^2(\Gamma\cup S,\mathbb{R}^2)$, from (5.16) and (5.18), the jump relations now yield

$$
\begin{aligned}
[\nu_S\cdot(\nabla u_1')-\nu_S\cdot(\nabla u_2')]&=p_{S_\nu}[k_1^2u_1-k_2^2u_2]+\nabla\cdot[p_{S_\nu}(\nabla_{S_\tau}u_1-\nabla_{S_\tau}u_2)]\\
&=p_{S_\nu}[k_1^2u_1-k_2^2u_2].
\end{aligned}
$$

Moreover, for every perturbation $\theta\in C^2(\Gamma\cup S,\mathbb{R}^2)$, the total fields and their normal derivative are assumed to satisfy

$$
\nu_{\Gamma_\theta}\cdot\nabla u_{1,\theta}=0 \quad \text{on } \Gamma_\theta.
$$

Hence,

$$
[\nu_{\Gamma_\theta}(I+\theta)]\cdot[(\nabla u_{1,\theta})(I+\theta)]=0 \quad \text{on } \Gamma. \qquad (5.19)
$$

It follows from (5.15) and (5.19) that

$$
\begin{aligned}
&[\nu_{\Gamma_\theta}(I+\theta)]\cdot[(\nabla u_{1,\theta})(I+\theta)]\\
=&\frac{1}{\|g(\theta)\nu_\Gamma\|_2}[g(\theta)\nu_\Gamma]\cdot[g(\theta)\nabla(u_{1,\theta}(I+\theta))]=0 \quad \text{on } \Gamma,
\end{aligned}
$$

which gives

$$
[g(\theta)\nu_\Gamma]\cdot[g(\theta)\nabla(u_{1,\theta}(I+\theta))]=0 \quad \text{on } \Gamma. \qquad (5.20)
$$

Using (5.20) and the sound hard boundary condition (2.7), we may follow the same steps as those for (5.18) and obtain

$$
\begin{aligned}
0&=\frac{\partial}{\partial\theta}\{[g(\theta)\nu_\Gamma]\cdot[g(\theta)\nabla(u_{1,\theta}(I+\theta))]\}(0)p\\
&=-\nabla\cdot[p_{\Gamma_\nu}(\nabla_{\Gamma_\tau}u_1)]-p_{\Gamma_\nu}(k_1^2u_1)+\nu_\Gamma\cdot(\nabla u_1') \quad \text{on } \Gamma,
\end{aligned}
$$

which yields that

$$
\nu_\Gamma\cdot(\nabla u_1')=k_1^2p_{\Gamma_\nu}u_1+\nabla\cdot[p_{\Gamma_\nu}(\nabla_{\Gamma_\tau}u_1)] \quad \text{on } \Gamma.
$$

Based on the existence of the domain derivatives u_j', the proof of the integral representations for u_j' follow in the same manner as for the integral representation of u_j. Therefore, the asymptotic behavior to the domain derivative u_j' has the same form as u_j. This means that the domain derivatives (u_1',u_2') are the radiating solutions of the problem (5.1). □

Introduce the domain $\Omega_{1,h}$ bounded by Γ_h and S_h, where

$$\Gamma_h = \{x + hq(x)\nu_\Gamma : x \in \Gamma\}, \quad S_h = \{x + hq(x)\nu_S : x \in S\}.$$

where $q \in C^2(\Gamma \cup S, \mathbb{R})$ and $h > 0$. For any two domains Ω_1 and $\Omega_{1,h}$ in $\mathbb{R}^2$, define the Hausdorff distance

$$\mathrm{dist}(\Omega_1, \Omega_{1,h}) = \max\{d(\Omega_{1,h}, \Omega_1), d(\Omega_1, \Omega_{1,h})\},$$

where

$$d(\Omega_1, \Omega_{1,h}) = \sup_{x \in \Omega_1} \inf_{y \in \Omega_{1,h}} |x - y|.$$

It can be easily seen that the Hausdorff distance between $\Omega_{1,h}$ and Ω_1 is of the order h, i.e., $\mathrm{dist}(\Omega_1, \Omega_{1,h}) = \mathcal{O}(h)$. We have the following local stability result.

Theorem 5.2. *If $q \in C^2(\Gamma \cup S, \mathbb{R})$ and $h > 0$ is sufficiently small, then*

$$\mathrm{dist}(\Omega_1, \Omega_{1,h}) \leq C \|u_{1,h} - u_1\|_{C^{1,\alpha}(\Gamma_1)},$$

where $u_{1,h}$ and u_1 is the solution of Problem 2.1 *corresponding to the domain $\Omega_{1,h}$ and Ω_1, respectively, and C is a positive constant independent of h.*

Proof. Assume by contradiction that there exists a subsequence from $\{u_{1,h}\}$, which is still denoted as $\{u_{1,h}\}$ for simplicity, such that

$$\lim_{h \to 0} \left\| \frac{u_{1,h} - u_1}{h} \right\|_{C^{1,\alpha}(\Gamma_1)} = \|u_1'\|_{C^{1,\alpha}(\Gamma_1)} = 0, \quad h \to 0,$$

which yields $u_1' = 0$ on Γ_1. Following a similar proof of Theorem 3.1, we can show the uniqueness of the solution for problem (5.1). An application of the uniqueness for problem (5.1) yields that $u_j' = 0$ in $\overline{\Omega}_j$, $j = 1,2$. Hence, $\nabla u_j' = 0$ in $\overline{\Omega}_j$, $j = 1,2$.

Taking $p = q(x)\nu_\Gamma$ on Γ in problem (5.1), we have from the boundary condition of u_1' in problem (5.1) that

$$\nu_\Gamma \cdot (\nabla u_1') = k_1^2 q u_1 + \nabla \cdot [q(\nabla_{\Gamma_\tau} u_1)] = 0.$$

Since q is arbitrary, we have $u_1 = 0$ on Γ, which is impossible by Lemma 4.2.

Consider the perturbation on S, take $p(x) = q(x)\nu_S$ on S in problem (5.1), from $u_j' = 0$ and $\nabla u_j' = 0$ in $\overline{\Omega}_j$, $j = 1,2$, one can get

$$\nu_S \cdot \nabla u_1' - \nu_S \cdot \nabla u_2' = 0 \quad \text{on } S.$$

One the other hand, from Theorem 5.1, one can get

$$(\nu_S \cdot \nabla u_1') - (\nu_S \cdot \nabla u_2') = p_{S_\nu}[k_1^2 u_1 - k_2^2 u_2] = q(k_1^2 - k_2^2)u_1 \quad \text{on } S.$$

For $q \neq 0$ and $k_1^2 \neq k_2^2$, it follows that $u_1 = 0$ on S, which is again impossible by Lemma 4.2. □

The result indicates that for small h, if the boundary measurements are $\mathcal{O}(h)$ close to each other, then the corresponding domains are also $\mathcal{O}(h)$ close to each other in the Hausdorff distance.

6 Conclusion

In this paper, we have studied the direct and inverse scattering problems of a point incident field by a sound hard obstacle which is immersed in a two-layered background medium separated by an infinite rough surface. The uniqueness of the solution for the direct scattering problem is proved. The existence of the solution for the direct scattering problem is discussed by using the method of boundary integral equations. For the inverse problem, we prove that the obstacle and the infinite rough surface are uniquely determined by the wave fields measured on two plane surfaces via a single point incident field. Moreover, we study the local stability of the inverse problem. It demonstrates that the Hausdorff distance of two domains is bounded above by the distance of the correspond wave fields if the two domains are close enough. A crucial step in the proof of the stability is to obtain the existence and characterization of the domain derivative of the wave field with respect to the change of the shape of the obstacle and the infinite rough surface. We deduce that the domain derivative satisfies a boundary value problem of the Helmholtz equation. The results are valid for the three-dimensional problem and the multiple obstacles which are located either above or below the infinite rough surface.

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References

[1] H. Ammari, G. Bao, and A. Wood, An integral equation method for the electromagnetic scattering from cavities, Math. Methods Appl. Sci., 23 (2000), 1057–1072.
[2] G. Bao, J. Gao, and P. Li, Analysis of direct and inverse cavity scattering problems, Numer. Math. Theor. Meth. Appl., 4 (2011), 419–442.
[3] G. Bao and J. Lin, Near-field imaging of the surface displacement on an infinite ground plane, Inverse Problems and Imaging, 7 (2013), 377–396.

[4] G. Bao and J. Lin, Imaging of local surface displacement on an infinite ground plane: the multiple frequency case, SIAM J. Appl. Math., 71 (2011), 1733–1752.
[5] G. Bao, L. Cowsar, and W. Masters, Mathematical Modeling in Optical Science, vol. 22, Frontiers in Applied Mathematics, SIAM, 2001.
[6] G. Bao and W. Sun, A fast algorithm for the electromagnetic scattering from a large cavity, SIAM J. Sci. Comput., 27 (2005), 553–574.
[7] G. Bao and P. Li, Near-field imaging of infinite rough surfaces in dielectric media, SIAM J. Imaging Sci., 7 (2014), 867–899.
[8] G. Bao and P. Li, Near-field imaging of infinite rough surfaces, SIAM J. Appl. Math., 73 (2013), 2162–2187.
[9] G. Bao and J. Lai, Optimal shape design of a cavity for radar cross section reduction, SIAM J. Control and Opt., 52 (2014), 2122–2140.
[10] G. Bao, T. Cui, and P. Li, Inverse diffraction grating of Maxwell's equations in biperiodic structures, Optics Express, 22 (2014), 4799–4816.
[11] G. Bao, A uniqueness theorem for an inverse problem in periodic diffractive optics, Inverse Problems, 10 (1994), 335–340.
[12] G. Bao and L. Zhang, Shape reconstruction of the multi-scale rough surface from multi-frequency phaseless data, Inverse Problems, 32 (2016), 1–16.
[13] G. Bao and L. Zhang, Uniqueness results for scattering and inverse scattering by infinite rough surfaces with tapered wave incidence, SIAM J. Imaging Sci., 11 (2018), 361–375.
[14] D. Colton and R. Kress, Integral Equation Methods in Scattering Theory, SIAM, Philadelphia, 2013.
[15] R. Djellouli and C. Farhat, On the Characterization of the Fréchet derivative with respect to a Lipschitz domain of the acoustic scattered field, J. Math. Anal. Appl., 238 (1999), 259–276.
[16] S. N. Chandler-Wilde and B. Zhang, A uniqueness result for scattering by infinite rough surfaces, SIAM J. Appl. Math., 58 (1998), 1774–1790.
[17] S. N. Chandler-Wilde and B. Zhang, Scattering of electromagnetic waves by rough interfaces and inhomogeneous layers, SIAM J. Math. Anal. 30 (1999), 559–583.
[18] S. N. Chandler-Wilde and P. Monk, Existence, uniqueness and variational methods for scattering by unbounded rough surfaces, SIAM J. Math. Anal., 37 (2005), 598–618.
[19] S. N. Chandler-Wilde, E. Heinemeyer, and R. Potthast, Acoustic scattering by mildly rough unbounded surfaces in three dimensions, SIAM J. Appl. Math., 66 (2006), 1002–1026.
[20] F. Hettlich, The domain derivative of time-harmonic electromagnetic waves at interfaces, Math. Methods Appl. Sci., 35 (2012), 1681–1689.
[21] Y. He, P. Li, and J. Shen, A new spectral method for numerical solution of the unbounded rough surface scattering problem, J. Comput. Phys., 275 (2014), 608–625.
[22] R. Hiptmair and J. Li, Shape derivatives for scattering problems, Inverse Problems, 34 (2018), 105001.
[23] A. Kirsch, The domain derivative and two applications in inverse scattering theory, Inverse Problems, 9 (1993), 81–96.
[24] R. Kress and T. Tran, Inverse scattering for a locally perturbed half-plane, Inverse Problems, 16 (2000), 1541–1559.
[25] R. Kress and W. Rundell, Inverse scattering for shape and impedance revisited, J. Integral Equations Applications, 30 (2018), 293–311.
[26] R. Kress and W. Rundell, Nonlinear integral equations and the iterative solution for an inverse boundary value problem, Inverse Problems, 21 (2005), 1207–1223.
[27] P. Li, H. Wu, and W. Zheng, Electromagnetic scattering by unbounded rough surfaces, SIAM

J. Appl. Math., 43 (2011), 1205–1231.
[28] P. Li, An inverse cavity problem for Maxwell's equations, J. Differential Equations, 252 (2012), 3209–3225.
[29] P. Li and J. Shen, Analysis of the scattering by an unbounded rough surface, Math. Methods Appl. Sci., 35 (2012), 2166–2184.
[30] P. Li, L.-L. Wang, and A. Wood, Analysis of transient electromagentic scattering from a three-dimensional open cavity, SIAM J. Appl. Math., 75 (2015), 1675–1699.
[31] P. Li, G. Zheng, and W. Zheng, Maxwell's equations in an unbounded structure, Math. Methods Appl. Sci., 40 (2016), 573–588.
[32] J. Li, G. Sun, and B. Zhang, The Kirsch–Kress method for inverse scattering by infinite locally rough interfaces, Applicable Analysis, 96 (2017), 85–107.
[33] J. Liu, G. Nakamura, and M. Sini, Reconstruction of the shape and surface impedance from acoustic scattering data for an arbitrary cylinder, SIAM J. Appl. Math. 67 (2007), 1124–1146.
[34] X. Liu, B. Zhang, and H. Zhang, A direct imaging method for inverse scattering by unbounded rough surfaces, SIAM J. Imaging Sci., 11 (2018), 1629–1650.
[35] Y. Lu and B. Zhang, Direct and inverse scattering problem by an unbounded rough interface with buried obstacles, arXiv:1610.03515v1, 2016.
[36] F. Murat and J. Simon, Sur le contrôle par un domaine géométrique, Rapport du L.A. 189, Universiteé Paris 6 (1974).
[37] J.-C. Nedelec and F. Starling, Integral equation methods in a quasi-periodic diffraction problem for the time-harmonic Maxwell's equations, SIAM J. Math. Anal., 22 (1991), 1679–1701.
[38] R. Potthast, Fréchet differentiability of boundary integral operators in inverse acoustic scattering, Inverse Problems, 10 (1994), 431–447.
[39] J. Simon, Differentiation with respect to the domain in boundary value problems, Numer. Funct. Anal. Optim., 2 (1980), 649–687.
[40] F. Qu, J. Yang, and B. Zhang, An approximate factorization method for inverse medium scattering with unknown buried objects, Inverse Problems, 33 (2017), 035007.
[41] B. Zhang and S. N. Chandler-Wilde, Integral equation methods for scattering by infinite rough surfaces, Math. Methods Appl. Sci., 26 (2003), 463–488.
[42] H. Zhang and B. Zhang, A novel integral equation for scattering by locally rough surfaces and application to the inverse problem, SIAM J. Appl. Math., 73 (2013), 1811–1829.
[43] L. Zhang, F. Ma, and J. Wang, Regularized conjugate gradient method with fast multipole acceleration for wave scattering from 1D fractal rough surface, Wave Motion, 50 (2013), 41–56.