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Numerical Analysis of the Midpoint Scheme for the Generalized Benjamin-Bona-Mahony Equation with White Noise Dispersion

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Abstract. We consider a midpoint scheme to approximate analytical solutions to a white noise driven BBM equation that reads

$$du - du_{xx} + u_x \circ dW + u^p u_x dt = 0.$$

We prove the well-posedness of the time-discrete approximation scheme and we provide the strong error order, which is 1.

AMS subject classifications: 35Q53, 65C30, 65N12, 65N15

Key words: Stochastic long wave equations, midpoint scheme, strong order of convergence.

1 Introduction

1.1 Dispersive equations with white noise modulation

The regularized long wave equation, also known as Benjamin-Bona-Mahony (BBM) equation has been introduced as an alternate model to Korteweg-de Vries equation for the propagation of one-way long wave in shallow water. For general nonlinearities, the equation (gBBM) reads

$$u_t - u_{txx} + u_x + u^p u_x = 0. \tag{1.1}$$

These deterministic equations have been widely studied theoretically and numerically in the mathematical literature; see for instance the initial value problem in [6,7,22], the decay rate of solutions for small initial data in [1,2,28] or numerical computations in [3,21].

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Recent years have seen an ever increasing study about stochastic dispersive equations, whether for theoretical [8–10,13–17,20] or numerical aspects [5,11,18,19,26]. Here we are interested in gBBM equations driven by white noise modulation. Quantities such as the bathymetry, the air pressure on the surface, or the wind effect, are stochastic in nature, it is therefore desirable to study stochastic shallow water wave equations [12]. We are dealing with the stochastic model studied in [8]. Here $x \in \mathbb{T}$ the one-dimension torus.

The study of dispersive equations with white noise modulation has latterly started with nonlinear Schrödinger equations (NLS), see [13,20]; NLS equations model also the propagation of water waves but in deep water. Such investigation furthers the study where the dispersion is driven by a deterministic varying function [4]. Since the dispersive solutions are oscillating, choosing the appropriate method to solve this type of stochastic PDE is of great interest, the approximation being generally of a lower order than in the deterministic case [24, 25]. Typically, time derivatives of dispersive PDEs are discretized by semi-implicit schemes [5], splitting methods [26], or exponential integrators [11]. These schemes have the advantage of solving the linear part unconditionally stable. Nonetheless, Euler-Maruyama scheme has been applied for the NLS equation with multiplicative noise in [14]. Authors showed that the strong error order, i.e. the mean-square order [24,25], is $1/2$, while the weak order, i.e. in the distribution sense [27,30], is 1.

In [5], the authors provide the strong error order for the Crank-Nicolson scheme approximating the NLS with white noise modulation. They proved that the strong order is 1 instead of 2 in the deterministic case. In this work we prove that for modulated stochastic BBM equation we also have a strong order 1 for a midpoint scheme; this is expected since, like the Crank-Nicolson scheme, the midpoint scheme has order 2 for deterministic ODE. It is worth to point out that the arguments used for NLS and for BBM are different.

The article is organized as follows. We complete the introduction reminding some important results about the generalized BBM equation with white noise modulation and we set the mathematical framework. In Section 2 the numerical scheme is presented and the main results are stated. In Sections 3 and 4, we provide the proofs of the main theorems. We discuss some numerical computations in Section 5.

1.2 A BBM equation with white noise modulation

We address the generalized BBM equation with white noise dispersion introduced in [8]. This equation reads, for p integer ≥ 1, $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the one-dimensional torus.

$$du - du_{xx} + u_x \circ dW + u^p u_x dt = 0 \tag{1.2}$$

in Stratonovich's formulation. This stands for the stochastic differential equation in $H^1(\mathbb{T})$

$$du + Au \circ dW + AF(u)dt = 0, \tag{1.3}$$

where A is the bounded skew symmetric operator $\partial_x(1-\partial_x^2)^{-1}$, the nonlinear term reads $F(u) = \frac{u^{p+1}}{p+1}$, and $W(t)$ is a standard real valued Brownian motion. Eq. (1.3) is a short-hand notation for the Ito's formulation that reads

$$du+AudW-\frac{1}{2}A^2udt+AF(u)dt=0. \tag{1.4}$$

Associated with this Brownian motion, there is a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and a stochastic basis $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\geq 0})$. The symbol $\circ$ stands for the Stratonovich's product. The initial value problem related to (1.2) is globally well-posed following the approach developed in [8].

Theorem 1.1. *Consider u_0 be in $L^2(\Omega,H^1(\mathbb{T}))$. There exist a unique solution u of (1.2) adapted to $\mathcal{F}_t$, with paths a.e. in $C([0,T),H^1(\mathbb{T}))$, for any $T>0$. Moreover the $H^1(\mathbb{T})$ norm of the solution is conserved along the paths, that is $||u(t)||_{H^1}=||u_0||_{H^1}$ almost surely. If moreover $\mathbb{E}(||u_0||^{2p+2}_{H^1(\mathbb{T})})<+\infty$, then u belongs to $L^2(\Omega;C([0,T),H^1(\mathbb{T}))$, for any $T>0$.*

1.3 Mathematical framework and notations

In the periodic setting we use the Fourier expansion (in space) of a function $v=\sum_{k\in\mathbb{Z}}\widehat{v}_k e_k$ where $e_k(x)=\exp(ikx)$.

Thanks to the Parseval equality, Sobolev norms are computed with this Fourier expansion as

$$||v||^2_{H^1(\mathbb{T})}=\langle v,v\rangle_{H^1(\mathbb{T})}=\sum_{k\in\mathbb{Z}}(1+k^2)|\widehat{v}_k|^2.$$

Throughout this article, we will use linear bounded operators L in $H^1(\mathbb{T})$ defined through their symbol $\widehat{L}(k)$ and acting as $Lu=\mathfrak{F}^{-1}(\widehat{L}(k)\widehat{u}_k)$ where $\mathfrak{F}^{-1}$ is the inverse Fourier transform. We denote the norm of L as a linear operator by $||L||_{\mathcal{L}(H^1(\mathbb{T}))}$.

We define the space of square integrable measurable functions

$$Z:=\{u;\forall s,u(s)\text{ is }\mathcal{F}_s\text{-measurable};u(s)\in L^2(\Omega;H^1(\mathbb{T})\},$$

whose norm reads $||u||^2_Z=\mathbb{E}\left(||u||^2_{H^1}\right)$ with the associated scalar product $\langle u,v\rangle_Z$.

For a linear operator $L(\delta W)$ that depends on $\delta W=W(t)-W(s)$, $t>s$, and whose symbol is $\widehat{L}(\delta W,k)$, we set for the sake of simplicity the operator $\mathbb{E}(L)$ whose symbol reads

$$\widehat{\mathbb{E}(L)}(k)=\mathbb{E}(\widehat{L})(k)=\frac{1}{\sqrt{2\pi(t-s)}}\int_{\mathbb{R}}\widehat{L}(y,k)\exp\Big(-\frac{y^2}{2(t-s)}\Big)dy. \tag{1.5}$$

The following statement will be used throughout this article.

Lemma 1.1. *Assume $u(s),v(s)$ be $\mathcal{F}_s$-measurable. Consider a bounded linear operator $L(\delta W)$. Then*

$$\langle L(\delta W)u,v\rangle_Z=\langle\mathbb{E}(L)u,v\rangle_Z,$$

and

$$||L(\delta W)u||^2_Z\leq\mathbb{E}\Big(||L(\delta W)||^2_{\mathcal{L}(H^1(\mathbb{T}))}\Big)||u||^2_Z.$$

Proof. By Parseval's equality

$$\langle L(\delta W)u,v\rangle_{H^1(\mathbb{T})} = \mathrm{Re}\sum_{k\in\mathbb{Z}}(1+k^2)\widehat{L}(\delta W,k)\widehat{u}_k\overline{\widehat{v}_k}. \tag{1.6}$$

Computing the expectation and using that $\widehat{L}(\delta W,k)$ is independent of $\widehat{u}_k\overline{\widehat{v}_k}$, we have from Fubini's theorem

$$\langle L(\delta W)u,v\rangle_Z = \mathbb{E}\left(\mathrm{Re}\sum_{k\in\mathbb{Z}}(1+|k|^2)\widehat{\mathbb{E}(L)}(k)\widehat{u}_k\overline{\widehat{v}_k}\right). \tag{1.7}$$

This completes the proof of the first assertion. The proof of the second inequality uses

$$||L(\delta W)u||_Z^2 \le \mathbb{E}\left(||L(\delta W)||^2_{\mathcal{L}(H^1(\mathbb{T}))}||u||^2_{H^1(\mathbb{T})}\right),$$

and the independence property. □

2 The time-discrete numerical scheme

To begin with, it is worth to observe that the choice of the midpoint discretization scheme lends itself to the very definition of the Stratonovich product. Indeed the integral $\int_0^T u(t)\circ dW_t$ is defined by the limit (as $N\to+\infty$) in probability of the sum

$$\sum_{n=0}^{N}\frac{u(t_n)+u(t_{n+1})}{2}(W_{t_{n+1}}-W_{t_n}).$$

2.1 A time discretization

Let $\delta t>0$ be a given time step, $t_n=n\delta t$, $\delta W_n=W(t_{n+1})-W(t_n)$. For a given (say deterministic for the sake of simplicity) initial data $u_0\in H^1(\mathbb{T})$ we solve by induction the following midpoint scheme

$$u_{n+1}-u_n+A\left(\frac{u_n+u_{n+1}}{2}\right)\delta W_n+A\left(F\left(\frac{u_n+u_{n+1}}{2}\right)\right)\delta t=0. \tag{2.1}$$

As shown below, the midpoint scheme is suitable to preserve the $H^1(\mathbb{T})$ norm of the solution along the trajectories. In the linear case ($F\equiv 0$), this scheme is related to the very definition of the Stratonovich integral. A drawback of this midpoint scheme is that it requires to solve a nonlinear equation to define the map $u_n\mapsto u_{n+1}$.

We now introduce the bounded linear operators

$$T_{n,\delta t}=\left(Id+\frac{\delta W_n}{2}A\right)^{-1},$$

and

$$S_{n,\delta t} = \left(Id + \frac{\delta W_n}{2} A\right)^{-1} \left(Id - \frac{\delta W_n}{2} A\right).$$

It is worth to point out that $S_{n,\delta t}$, which has symbol $\left(1+\frac{i\delta W_n a(k)}{2}\right)^{-1}\left(1-\frac{i\delta W_n a(k)}{2}\right)$, with $a(k)=k(1+k^2)^{-1}$, is a unitary operator in $H^1(\mathbb{T})$. Then the scheme (2.1) reads also

$$u_{n+1} = S_{n,\delta t} u_n - \delta t T_{n,\delta t} A\left(F\left(\frac{u_n+u_{n+1}}{2}\right)\right). \tag{2.2}$$

The well-posedness of the scheme is given by the following proposition.

Proposition 2.1. Let $T>0$ be arbitrary, let $u_0 \in H^1(\mathbb{T})$ be a deterministic initial datum. There exist $N \in \mathbb{N}$ (large enough) and $\delta t = \frac{T}{N}$ depending on $||u_0||_{H^1(\mathbb{T})}$ such that there exists a unique discrete solution $(u_n)_{n\in\mathbb{N}}$ satisfying (2.1) for all $n \le N$, which belongs almost surely in to $H^1(\mathbb{T})$ and that is $\mathcal{F}_{t_n}$-measurable. Moreover the $H^1(\mathbb{T})$ norm of the solution is preserved, that is for any $n \ge 0$, almost surely $||u_n||_{H^1(\mathbb{T})} = ||u_0||_{H^1(\mathbb{T})}$.

We now state that the strong order of the convergence in probability of the scheme is 1.

Theorem 2.1. *Under the assumption of Proposition* 2.1*, for $n \le N$,*

$$\lim_{C\to\infty} \mathbb{P}\left(\sup_{\delta t>0} \frac{||u(t_n)-u_n||_{H^1(\mathbb{T})}}{\delta t} \ge C\right) = 0.$$

Here the supremum is understood for δt in $(0,\varepsilon_0]$ for ε_0 small enough. The proofs are postponed respectively to Section 3 and Section 4.

3 Proof of Proposition 2.1

We rather prove the existence of $\tilde{u} = \frac{u_n+u_{n+1}}{2}$ that is solution to

$$\widetilde{u} + \frac{1}{2} A\widetilde{u}\delta W_n + \frac{1}{2} AF(\widetilde{u})\delta t - u_n = 0, \tag{3.1}$$

and then set $u_{n+1} = 2\widetilde{u} - u_n$.

Fix $R = ||u_0||_{H^1(\mathbb{T})}$ and introduce the closed subset of Z

$$Z_R = \{v \in Z; ||v||_{H^1(\mathbb{T})} \le 2R \ a.s.\}.$$

We seek a fixed point in Z_R for the map

$$\begin{aligned} \mathcal{T}_{u_n} : H^1(\mathbb{T}) &\longrightarrow H^1(\mathbb{T}) \\ v &\longmapsto \left(1+\frac{\delta W_n}{2} A\right)^{-1} \left(u_n - \frac{\delta t}{2} AF(v)\right). \end{aligned}$$

Using that $H^1(\mathbb{T})$ is a Banach algebra that is embedded into $L^\infty(\mathbb{T})$, we know that

$$||F(u)-F(v)||_{H^1(\mathbb{T})} \leq c(||u||^p_{H^1(\mathbb{T})}+||v||^p_{H^1(\mathbb{T})})||u-v||_{H^1(\mathbb{T})}. \tag{3.2}$$

Then, for any $v_1,v_2 \in Z_R$,

$$||F(v_1)-F(v_2)||_Z \leq c\delta t(2R)^p||v_1-v_2||_Z. \tag{3.3}$$

Then for $\delta t R^p \ll 1$, since $AT_{n,\delta t}$ has norm less than 1, the map $\mathcal{T}_{u_n}$ is contracting. To prove that the map send Z_R into Z_R if δt is small enough is similar and then omitted. Moreover, since the map $u_n \mapsto \mathcal{T}_{u_n}$ is linear one can prove that the map $u_n \mapsto \tilde{u}$ (the fixed point of $\mathcal{T}_{u_n}$) is continuous. It is then standard to prove that u_{n+1} is $\mathcal{F}_{t_{n+1}}$-measurable.

It remains to prove that the $H^1(\mathbb{T})$ norm of u_n is preserved. Consider the scalar product of Eq. (2.1) with $u_{n+1}+u_n$ in $H^1(\mathbb{T})$, it leads to

$$\begin{aligned} ||u_{n+1}||^2_{H^1(\mathbb{T})} - ||u_n||^2_{H^1(\mathbb{T})} + \frac{\delta W_n}{2}\langle A(u_{n+1}+u_n), u_{n+1}+u_n\rangle_{H^1(\mathbb{T})} \\ + \quad 2\delta t \langle AF\left(\frac{u_n+u_{n+1}}{2}\right), \frac{u_{n+1}+u_n}{2}\rangle_{H^1(\mathbb{T})} = 0. \end{aligned}$$

Since the operator A is skew symmetric then the third term in the left hand side vanishes. Besides $\langle AF(v),v\rangle_{H^1(\mathbb{T})} = \int_{\mathbb{T}}(F(v))_x v dx = 0$, then $||u_{n+1}||^2_{H^1(\mathbb{T})} = ||u_n||^2_{H^1(\mathbb{T})}$. This completes the proof of the proposition.

4 Proof of the main theorem

The proof of the main theorem relies on the following statement

Theorem 4.1. *Let $T>0$ and $u_0 \in H^1(\mathbb{T})$. Let u be the unique solution of* (1.2) *in $L^2(\Omega;C([0,T];H^1(\mathbb{T})))$ and u_n the discrete solution of* (2.1) *at time $t_n = n\delta t$. Then there exists a constant $C>0$ that depends on u_0,T, but that is independent of δt, such that for all $n \leq N$,*

$$||u(t_n)-u_n||^2_Z \leq C\delta t^2.$$

Theorem 2.1 is then a mere consequence of Theorem 4.1 using Markov's inequality.

We divide the proof of Theorem 4.1 as follows. We first compute the modulus of continuity of the analytic solution. Then we provide an upper bound of the consistency error. Eventually we prove the error estimate.

4.1 Upper bound for the modulus of continuity

Let $u(t)$ be the process that is a mild solution to Eq. (1.2) and that reads

$$u(t) = S(t,s)u(s) - \int_s^t S(t,\tau)AF(u(\tau))d\tau, \tag{4.1}$$

where A is the bounded skew symmetric operator whose symbol is $ia(k)=ik(1+k^2)^{-1}$, $S(t,s)$ denotes the linear unitary operator whose symbol is $\exp(-ia(k)(W(t)-W(s)))$ and where $F(u)=\frac{u^{p+1}}{p+1}$. We refer in the sequel this process as the analytic solution.

For $0<t-s<\delta t$, we define the modulus of continuity $\theta(t,s)$ of the analytic solution as

$$\theta(t,s)=u(t)-u(s)=(S(t,s)-Id)u(s)-\int_s^t S(t,\tau)AF(u(\tau))d\tau=\lambda+\mu, \tag{4.2}$$

where $\lambda=(S(t,s)-Id)u(s)$ denotes the linear part and μ denotes the remaining term. We first prove.

Lemma 4.1. *There exists a constant $C(u_0)>0$ that depends on the $H^1(\mathbb{T})$ norm of u_0 such that*

$$(t-s)||\lambda||_Z^2+||\mu||_Z^2+||\mathbb{E}(\lambda)||_{H^1(\mathbb{T})}^2\leq C(u_0)(t-s)^2. \tag{4.3}$$

Therefore

$$||\theta(t,s)||_Z^2\leq C(u_0)(t-s). \tag{4.4}$$

Remark 4.1. The constant $C(u_0)$ may vary from one line to one another without notice. Similarly c denotes a numerical constant independent of n that may also vary. Inequality (4.3) shows that the remaining term μ is a lower order term with respect to λ, but compares in some sense with $\mathbb{E}(\lambda)$.

Proof. Let us observe that since A, $S(t,s)$ are bounded operators and since $H^1(\mathbb{T})$ is a Banach algebra then

$$||\mu||_{H^1(\mathbb{T})}\leq c\int_s^t ||u(\tau)||_{H^1(\mathbb{T})}^{p+1}d\tau. \tag{4.5}$$

Appealing Theorem 1.1 that asserts that a.s. the $H^1(\mathbb{T})$ norm of the analytical solution is constant along the trajectory, we infer taking the expectation of the square of (4.5) that $||\mu||_Z\leq C(u_0)(t-s)$.

We now focus on the linear part. By Parseval's equality

$$||\lambda||_{H^1(\mathbb{T})}^2=\sum_{k\in\mathbb{Z}}(1+k^2)|\widehat{S}(t,s)_k-1|^2|\widehat{u}_k(s)|^2, \tag{4.6}$$

where $\widehat{S}(t,s)$ is the symbol of the linear unitary operator. Therefore

$$||\lambda||_{H^1(\mathbb{T})}^2\leq||\widehat{S}(t,s)-1||_{l^\infty(\mathbb{Z})}^2||u(s)||_{H^1(\mathbb{T})}^2, \tag{4.7}$$

where $||\widehat{S}(t,s)-1||_{l^\infty(\mathbb{Z})}=\sup_k|(\widehat{S}(t,s)-1)(k)|$. We observe that $||\widehat{S}(t,s)-1||_{l^\infty(\mathbb{Z})}^2\leq c|W(t)-W(s)|^2$. Then using once again that $||u(s)||_{H^1(\mathbb{T})}^2=||u_0||_{H^1(\mathbb{T})}^2$ and that $\widehat{S}(t,s)-1$ is independent of $u(s)$ $\mathcal{F}_s$-measurable, the expectation of (4.7) gives

$$||\lambda||_Z^2\leq C(u_0)\mathbb{E}(|W(t)-W(s)|^2)=C(u_0)(t-s). \tag{4.8}$$

This completes the proof of (4.4). It remains to bound by above the $H^1(\mathbb{T})$ norm of the expectation of λ. Using that $\widehat{S}(t,s)-1$ is independent of $u(s)$, the inverse Fourier transform provides

$$\mathbb{E}(\lambda)=\mathfrak{F}^{-1}\Big(\mathbb{E}(\widehat{S}(t,s)-1)\mathbb{E}(\widehat{u}(s))\Big). \tag{4.9}$$

Since $W(t)-W(s)\sim\mathcal{N}(0,t-s)$, then

$$\mathbb{E}(\widehat{S}(t,s)-1)(k)=\exp\left(-\frac{(t-s)a(k)^2}{2}\right)-1, \tag{4.10}$$

and $||\mathbb{E}(\widehat{S}(t,s)-1)||_{l^\infty(\mathbb{Z})}\leq c(t-s)$. We deduce from (4.9) and from Parseval's identity that

$$||\mathbb{E}(\lambda)||_{H^1(\mathbb{T})}\leq c(t-s)||u(s)||_Z=C(u_0)(t-s), \tag{4.11}$$

since the $H^1(\mathbb{T})$ norm of the trajectory is a.s. constant along the path. □

4.2 Consistency error

Consider $u(t)$ the analytical solution of (1.2). Let us define the consistency error ϵ_n at time $n\delta t$ as

$$\epsilon_n=u(t_{n+1})-S_{n,\delta t}u(t_n)+\delta t T_{n,\delta t}A\left(F\left(\frac{u(t_n)+u(t_{n+1})}{2}\right)\right). \tag{4.12}$$

Proposition 4.1. There exists a constant $C(u_0)>0$ that depends on the H^1 norm of u_0 such that

$$||\epsilon_n||_Z\leq C(u_0)(\delta t)^{\frac{3}{2}}. \tag{4.13}$$

Proof. Using that $u(t)$ solves (4.1), we expand the consistency error as

$$\epsilon_n=\epsilon_n^1+\epsilon_n^2+\epsilon_n^3+\epsilon_n^4, \tag{4.14}$$

where

- $\epsilon_n^1=(S(t_{n+1},t_n)-S_{n,\delta t})u(t_n)$,
- $\epsilon_n^2=\displaystyle\int_{t_n}^{t_{n+1}}S(t_{n+1},s)A\left(F\left(\frac{u(t_n)+u(t_{n+1})}{2}\right)-F(u(s))\right)ds$,
- $\epsilon_n^3=\delta t(T_{n,\delta t}-Id)A\left(F\left(\frac{u(t_n)+u(t_{n+1})}{2}\right)\right)$,
- $\epsilon_n^4=\left(\displaystyle\int_{t_n}^{t_{n+1}}(Id-S(t_{n+1},s))ds\right)A\left(F\left(\frac{u(t_n)+u(t_{n+1})}{2}\right)\right)$.

As for (4.6)-(4.7), Parseval's inequality is appealed for the first term to get

$$||\epsilon_n^1||_{H^1(\mathbb{T})} \leq ||\widehat{S}(t_{n+1},t_n) - \widehat{S}_{n,\delta t}||_{l^\infty(\mathbb{Z})}||u(t_n)||_{H^1(\mathbb{T})}. \tag{4.15}$$

We have the identity

$$(\widehat{S}(t_{n+1},t_n) - \widehat{S}_{n,\delta t})(k) = \exp(-ia(k)\delta W_n) - \exp\left(-2i\arctan\left(\frac{a(k)\delta W_n}{2}\right)\right). \tag{4.16}$$

and because $a(k) = \frac{k}{1+k^2}$, $||\widehat{S}(t_{n+1},t_n) - \widehat{S}_{n,\delta t}||_{l^\infty(\mathbb{Z})} \leq c|\delta W_n|^3$. Gathering this with (4.15) and taking the expectation we then have

$$||\epsilon_n^1||_Z^2 \leq c\mathbb{E}\left(|\delta W_n|^6 ||u(t_n)||^2_{H^1(\mathbb{T})}\right). \tag{4.17}$$

Since $||u(t_n)||^2_{H^1(\mathbb{T})}$ is $\mathcal{F}_{n\delta t}$-measurable then using the independence and the conservation of the $H^1(\mathbb{T})$ norm,

$$||\epsilon_n^1||_Z^2 \leq c\mathbb{E}(|\delta W_n|^6)\mathbb{E}(||u(t_n)||^2_{H^1(\mathbb{T})}) = c(\delta t)^3 ||u_0||^2_{H^1(\mathbb{T})}. \tag{4.18}$$

We now tackle the second term. Appealing (3.2), using that $S_{n,\delta t}$ is unitary, the fact that $u \mapsto AF(u)$ is locally Lipschitz and the conservation of the $H^1(\mathbb{T})$ norm, we have

$$||\epsilon_n^2||^2_{H^1(\mathbb{T})} \leq c(\delta t)\int_{t_n}^{t_{n+1}} ||u_0||^{2p}_{H^1(\mathbb{T})}\left(||u(s)-u(t_n)||^2_{H^1(\mathbb{T})} + ||u(s)-u(t_{n+1})||^2_{H^1(\mathbb{T})}\right)ds. \tag{4.19}$$

Lemma 4.1 provides

$$||\epsilon_n^2||_Z^2 \leq C(u_0)(\delta t)\int_{t_n}^{t_{n+1}} (\delta t)ds = C(u_0)(\delta t)^3.$$

We proceed as above for the third term. Using that $H^1(\mathbb{T})$ is an algebra and the conservation of the $H^1(\mathbb{T})$ norm of the analytic solution

$$||\epsilon_n^3||_{H^1(\mathbb{T})} \leq \delta t ||\widehat{T}_{n,\delta t} - 1||_{l^\infty(\mathbb{Z})} \left|\left|A\left(F\left(\frac{u(t_n)+u(t_{n+1})}{2}\right)\right)\right|\right|_{H^1(\mathbb{T})} \leq C(u_0)\delta t |\delta W_n|. \tag{4.20}$$

Therefore $||\epsilon_n^3||_Z^2 \leq C(u_0)(\delta t)^3$. The bound of the term ϵ_n^4 is similar and then omitted. □

4.3 Computing the error estimate

Defining the error at time t_n by $e_n = u(t_n) - u_n$, the propagation of the error reads as

$$e_{n+1} = S_{n,\delta t}e_n + \delta t T_{n,\delta t} A\left(F\left(\frac{u_n + u_{n+1}}{2}\right) - F\left(\frac{u(t_n)+u(t_{n+1})}{2}\right)\right) + \epsilon_n. \tag{4.21}$$

We prove:

Lemma 4.2. *There exists two positive constants $c(u_0)$,$C(u_0)$ such that*

$$(1-c(u_0)\delta t)||e_{n+1}||_Z^2 \leq (1+c(u_0)\delta t)||e_n||_Z^2 + C(u_0)(\delta t)^3.$$

If this lemma is granted, the proof of Theorem 4.1 is a direct consequence of the discrete Gronwall lemma. It remains then to prove Lemma 4.2.

Proof. Splitting $e_{n+1} = e^1_{n+1} + e^2_{n+1} + e^3_{n+1}$ accordingly to (4.21), we have

$$||e_{n+1}||_Z^2 = \sum_j ||e^j_{n+1}||_Z^2 + 2\sum_{i<j} \langle e^i_{n+1}, e^j_{n+1}\rangle_Z. \tag{4.22}$$

We start with the three diagonal terms. Since $S_{n,\delta t}$ is a unitary operator, then

$$||e^1_{n+1}||_Z^2 = ||e_n||_Z^2. \tag{4.23}$$

Then, since $T_{n,\delta t}$ and A are bounded operators, whose norm bounded by 1, and since $||u_n||_{H^1(\mathbb{T})} = ||u(t_n)||_{H^1(\mathbb{T})} = ||u_0||_{H^1(\mathbb{T})}$, inequality (3.2) implies

$$||e^2_{n+1}||_Z^2 \leq C(u_0)(\delta t)^2 (||e_{n+1}||_Z^2 + ||e_n||_Z^2). \tag{4.24}$$

Besides, from Lemma 4.1 we know that $||e^3_{n+1}||_Z^2 \leq C(u_0)(\delta t)^3$.

We now handle the off-diagonal terms. From the Cauchy-Schwarz inequality, it comes

$$2|\langle e^2_{n+1}, e^3_{n+1}\rangle_Z| \leq ||e^2_{n+1}||_Z^2 + ||e^3_{n+1}||_Z^2, \tag{4.25}$$

and analogously

$$|\langle e^1_{n+1}, e^2_{n+1}\rangle_Z| \leq ||e^1_{n+1}||_Z ||e^2_{n+1}||_Z \leq C(u_0)(\delta t)(||e_{n+1}||_Z^2 + ||e_n||_Z^2). \tag{4.26}$$

To tackle the last term, we introduce $e^1_{n+1} := S_{n,\delta t} e_n = e_n + (S_{n,\delta t} - Id)e_n$, thus

$$\langle e^1_{n+1}, e^3_{n+1}\rangle_Z = \langle e_n, \epsilon_n\rangle_Z + \langle (S_{n,\delta t} - Id)e_n, \epsilon_n\rangle_Z.$$

We now bound by above the second term in the right hand side of this equality. By Cauchy-Schwarz inequality, we have

$$|\langle (Id - S_{n,\delta t})e_n, \epsilon_n\rangle_Z| \leq ||(Id - S_{n,\delta t})e_n||_Z ||\epsilon_n||_Z. \tag{4.27}$$

On the one hand, Lemma 4.1 gives $||\epsilon_n||_Z \leq C(u_0)(\delta t)^{\frac{3}{2}}$. On the other hand, proceeding as in (4.6)-(4.7), one gets

$$||(Id - S_{n,\delta t})e_n||_Z^2 \leq \mathbb{E}(||1 - \widehat{S}_{n,\delta t}||^2_{l^\infty(\mathbb{Z})} ||e_n||^2_{H^1(\mathbb{T})}), \tag{4.28}$$

and using that $||e_n||^2_{H^1}$ is $\mathcal{F}_{t_n}$ measurable

$$\mathbb{E}(||1 - \widehat{S}_{n,\delta t}||^2_{l^\infty(\mathbb{Z})} ||e_n||^2_{H^1(\mathbb{T})}) = \mathbb{E}(||1 - \widehat{S}_{n,\delta t}||^2_{l^\infty(\mathbb{Z})}) ||e_n||_Z^2.$$

Since $\mathbb{E}(||1-\widehat{S}_{n,\delta t}||^2_{l^\infty(\mathbb{Z})})\leq c\mathbb{E}(|\delta W_n|^2)=c\delta t$, then we have that

$$|((Id-S_{n,\delta t})e_n,\epsilon_n)_Z|\leq C(u_0)(\delta t)^2||e_n||_Z.$$

We now focus on the remaining term $(e_n,\epsilon_n)_Z$. We split ϵ_n according to (4.14). Appealing the Parseval theorem and using that both e_n and $u(t_n)$ are $\mathcal{F}_{t_n}$ measurable, we have from Lemma 1.1

$$\langle e_n,\epsilon_n^1\rangle_Z=\mathbb{E}\left(\mathrm{Re}\sum_{k\in\mathbb{Z}}(1+|k|^2)\overline{\widehat{e_n}_k}\mathbb{E}(\widehat{S}(t_{n+1},t_n)-\widehat{S}_{n,\delta t})(k)\widehat{u(t_n)}_k\right), \tag{4.29}$$

and independence implies

$$|\langle e_n,\epsilon_n^1\rangle_Z|\leq||e_n||_Z||\mathbb{E}(\widehat{S}(t_{n+1},t_n)-\widehat{S}_{n,\delta t})||_{l^\infty(\mathbb{Z})}||u(t_n)||_Z. \tag{4.30}$$

This is bounded by $C(u_0)(\delta t)^2||e_n||_Z$ since the $H^1(\mathbb{T})$ norm of the analytic solution is a.s. constant and due to the following estimate

$$||\mathbb{E}(\widehat{S}(t_{n+1},t_n)-\widehat{S}_{n,\delta t})||_{l^\infty(\mathbb{Z})}\leq c(\delta t)^2. \tag{4.31}$$

Inequality (4.31) is true because $\delta W_n\sim\mathcal{N}(0,\delta t)$ and

$$\mathbb{E}(\widehat{S}(t_{n+1},t_n)-\widehat{S}_{n,\delta t})(k)=\frac{1}{\sqrt{2\pi\delta t}}\int_{\mathbb{R}}\exp\left(-\frac{y^2}{2\delta t}\right)\left(\exp(-ia(k)y)-\frac{2-iya(k)}{2+iya(k)}\right)dy. \tag{4.32}$$

Using Taylor's formula, there exists two positive constant c,C such that

$$\left|\exp(-ia(k)y)-\frac{2-iya(k)}{2+iya(k)}-c\big(ya(k)\big)^3\right|\leq Cy^4a(k)^4. \tag{4.33}$$

Since $\int_{\mathbb{R}}\exp(-\frac{y^2}{2\delta t})y^3dy=0$, then

$$||\mathbb{E}(\widehat{S}(t_{n+1},t_n)-\widehat{S}_{n,\delta t})||_{l^\infty(\mathbb{Z})}\leq C\int_{\mathbb{R}}\exp(-\frac{y^2}{2\delta t})y^4\frac{dy}{\sqrt{\delta t}},$$

and the estimate (4.31) is proved.

We now tackle the second term. We first split ϵ_n^2 introducing artificially $F(u(t_n))$

$$\begin{aligned}\epsilon_n^2&=\epsilon_n^{21}+\epsilon_n^{22}+\epsilon_n^{23}\\&=\left(\int_{t_n}^{t_{n+1}}(S(t_{n+1},s)-Id)A\left(F\left(\frac{u(t_n)+u(t_{n+1})}{2}\right)-F(u(s))\right)ds\right)\\&\quad+\left(\int_{t_n}^{t_{n+1}}A(F(u(t_n)-F(u(s))))ds\right)+\left(\delta tA\left(F\left(\frac{u(t_n)+u(t_{n+1})}{2}\right)-F(u(t_n))\right)\right).\end{aligned} \tag{4.34}$$

We first have, introducing the adjoint operator $S^*(t_{n+1},s)$ of $S(t_{n+1},s)$

$$|\langle e_n,\epsilon_n^{21}\rangle_Z| \leq \int_{t_n}^{t_{n+1}} ||(S^*(t_{n+1},s)-Id)e_n||_Z \left|\left|A\left(F\left(\frac{u(t_n)+u(t_{n+1})}{2}\right)-F(u(s))\right)\right|\right|_Z ds. \tag{4.35}$$

Proceeding as in (4.6)-(4.7), we first have

$$||(S^*(t_{n+1},s)-Id)e_n||_Z \leq \mathbb{E}(||\widehat{S}(t_{n+1},s)-1||_{l^\infty(\mathbb{Z})})||e_n||_Z \leq c(\delta t)^{\frac{1}{2}}||e_n||_Z. \tag{4.36}$$

We also use (3.2) and Lemma 4.1 to bound $||A(F(\frac{u(t_n)+u(t_{n+1})}{2})-F(u(s)))||_Z$ by $C(u_0)(\delta t)^{\frac{1}{2}}$. Gathering these inequalities and integrating in time lead to

$$|\langle e_n,\epsilon_n^{21}\rangle_Z| \leq C(u_0)(\delta t)^2||e_n||_Z.$$

We now bound the term involving ϵ_n^{22}. The idea is to rewrite $\epsilon_n^{22}=\lambda_n^{22}+\mu_n^{22}$ analogously to (4.2). For this purpose, thanks to Taylor's formula (and uniform bounds in $H^1(\mathbb{T})$ on the analytic solution $u(t)$), and introducing DF the differential of F, we have

$$\epsilon_n^{22} = A\int_{t_n}^{t_{n+1}} DF(u(t_n))(u(t_n)-u(s))ds+\rho_n, \tag{4.37}$$

where almost surely for $s\in(t_n,t_{n+1})$

$$|\rho_n| \leq C(u_0)(\delta t)|u(t_n)-u(s)|^2.$$

Appealing the splitting of $u(s)-u(t_n)=(S(s,t_n)-Id)u(t_n)-\int_{t_n}^s S(s,\tau)AF(u(\tau))d\tau =: \lambda+\mu$, we then set

$$\lambda_n^{22} = -A\int_{t_n}^{t_{n+1}} DF(u(t_n))\lambda ds, \quad \text{and} \quad \mu_n^{22} = -A\int_{t_n}^{t_{n+1}} DF(u(t_n))\mu ds+\rho_n.$$

Lemma 4.1 provides

$$|\langle e_n,\rho_n\rangle_Z| \leq C(u_0)(\delta t)||e_n||_Z||u(t_n)-u(s))||_Z^2 \leq C(u_0)(\delta t)^2||e_n||_Z, \tag{4.38}$$

and

$$|\int_{t_n}^{t_{n+1}} \langle e_n,A\mu\rangle_Z ds| \leq (\delta t)||e_n||_Z \sup_s ||\mu||_Z \leq C(u_0)(\delta t)^2||e_n||_Z. \tag{4.39}$$

Since e_n and $u(t_n)$ are $\mathcal{F}_{t_n}$ measurable, Lemma 1.1 leads to

$$\left\langle e_n,A\int_{t_n}^{t_{n+1}} DF(u(t_n))\lambda ds\right\rangle_Z = \left\langle e_n,A\int_{t_n}^{t_{n+1}} DF(u(t_n))(\mathbb{E}(S(s,t_n)-Id)u(t_n))ds\right\rangle_Z. \tag{4.40}$$

Therefore, using the conservation of the $H^1(\mathbb{T})$ norm and (4.7)-(4.8), it comes

$$\begin{aligned}\left|\left\langle e_n,A\int_{t_n}^{t_{n+1}} DF(u(t_n))\lambda ds\right\rangle_Z\right| &\leq C(u_0)||e_n||_Z \int_{t_n}^{t_{n+1}} ||\mathbb{E}(\widehat{S}(s,t_n)-1)||_{l^\infty(\mathbb{Z})}ds||u(t_n)||_Z \\ &\leq C(u_0)(\delta t)^2||e_n||_Z. \end{aligned}\tag{4.41}$$

We deal similarly to bound by above ϵ_n^{23} and the proof is omitted.

We now focus on the term involving ϵ_n^3. We first split as above

$$\begin{aligned}\epsilon_n^3 &= \epsilon_n^{31}+\epsilon_n^{32}\\ &= \delta t(T_{n,\delta t}-Id)A\big(F(u(t_n))\big)+\delta t(T_{n,\delta t}-Id)A\left(F\left(\frac{u(t_n)+u(t_{n+1})}{2}\right)-F(u(t_n))\right).\end{aligned} \tag{4.42}$$

As for the upper bound (4.35), one gets

$$\begin{aligned}|\langle e_n,\epsilon_n^{32}\rangle_Z| &\le \delta t\mathbb{E}\left(||\widehat{T}_{n,\delta t}-1||_{l^\infty(\mathbb{Z})}||e_n||_{H^1}\right)\left|\left|A\left(F\left(\frac{u(t_n)+u(t_{n+1})}{2}\right)-F(u(t_n))\right)\right|\right|_Z\\ &\le \left(c(\delta t)^{\frac{3}{2}}||e_n||_Z\right)\left(C(u_0)(\delta t)^{\frac{1}{2}}\right).\end{aligned} \tag{4.43}$$

On the other hand, since e_n and $F(u(t_n))$ are $\mathcal{F}_{t_n}$ measurable, Lemma 1.1 implies

$$\delta t\langle e_n,(T_{n,\delta t}-Id)AF(u(t_n))\rangle_Z = \delta t\langle e_n,\mathbb{E}(T_{n,\delta t}-Id)AF(u(t_n))\rangle_Z. \tag{4.44}$$

Therefore proceeding as in (4.41)

$$|\delta t\langle e_n,(T_{n,\delta t}-Id)AF(u(t_n))\rangle_Z| \le C(u_0)(\delta t)||e_n||_Z||\widehat{\mathbb{E}(T_{n,\delta t})}-1||_{l^\infty(\mathbb{Z})}, \tag{4.45}$$

and we conclude using

$$(\widehat{\mathbb{E}(T_{n,\delta t})}-1)(k)=\frac{1}{\sqrt{2\pi\delta t}}\int_{\mathbb{R}}\exp\left(-\frac{y^2}{2\delta t}\right)\left(\left(1+\frac{ia(k)y}{2}\right)^{-1}-1\right)dy. \tag{4.46}$$

Since we have the existence of $c>0$ such that

$$\left|\left(1+\frac{ia(k)y}{2}\right)^{-1}-1+\frac{ia(k)y}{2}\right| \le c\frac{a(k)^2y^2}{4}, \tag{4.47}$$

and $\int_{\mathbb{R}}\exp\left(-\frac{y^2}{2\delta t}\right)ydy=0$, then $||\widehat{\mathbb{E}(T_{n,\delta t})}-1||_{l^\infty(\mathbb{Z})}\le c\delta t$.

The proof for the term ϵ_n^4 being similar, it is omitted for the sake of conciseness. □

5 Numerical simulations

5.1 Space and time discretizations

The space discretization relies on a standard Fast Fourier Transform (FFT), as detailed in [8]. The interval $[-\pi,\pi]$ is discretized into N_x points and for any vector u, its FFT is denoted by $\widehat{u}=(\widehat{u}_{-\frac{N_x}{2}},\cdots,\widehat{u}_{\frac{N_x}{2}-1})$ again, where $\widehat{u}_k$ is the k-th Fourier mode of u defined as

$$\widehat{u}_k=\sum_{j=-\frac{N_x}{2}}^{\frac{N_x}{2}-1}u_je^{-ijk}.$$

The Fourier transform in space is applied to the time-discrete scheme (2.1) to get for $-\frac{N_x}{2} \le k \le \frac{N_x}{2} - 1$,

$$(\widehat{u_{n+1}})_k - (\widehat{u_n})_k + \frac{ik}{1+k^2}\left(\widehat{\frac{u_n+u_{n+1}}{2}}\right)_k \delta W_n + \frac{ik}{1+k^2}\frac{1}{p+1}\left(\widehat{\left(\frac{u_n+u_{n+1}}{2}\right)^{p+1}}\right)_k \delta t = 0.$$

In accordance with the theoretical part, a buffering variable v is introduced as follows

$$\begin{cases} \widehat{v}_k + \frac{1}{2}\frac{ik}{1+k^2}\widehat{v}_k \delta W_n + \frac{1}{2}\frac{ik}{1+k^2}\frac{(\widehat{v^{p+1}})_k}{p+1}\delta t - (\widehat{u_n})_k = 0, \\ u_{n+1} = 2v - u_n. \end{cases}$$

This nonlinear equation is rewritten as the fixed point problem

$$\widehat{v}_k = \frac{2(1+k^2)(\widehat{u_n})_k - ik\delta t \frac{(\widehat{v^{p+1}})_k}{p+1}}{2(1+k^2) + ik\delta W_n}$$

solved thanks to Algorithm 1.

Algorithm 1: midpoint with Picard fixed-point scheme

Given an initial datum u_0, $\tau > 0$ a fixed tolerance and $N \in \mathbb{N}^*$, $M \in \mathbb{N}^*$ maximum numbers of iterations.

for $n = 1, 2, \cdots N$ **do**

$\delta W_n \sim \mathcal{N}(0, \sqrt{\delta t})$

$\widehat{v}_0 = \widehat{u_n}$

while $||v_{m+1} - v_m||_{H^1} > \tau ||v_0||_{H^1}$ and $m \le M$ **do**

$$\widehat{v}_{m+1,k} := \frac{2(1+k^2)(\widehat{u_n})_k - ik\delta t \frac{(\widehat{v_m^{p+1}})_k}{p+1}}{2(1+k^2) + ik\delta W_n}$$

$v = v_{m+1}$

$u_{n+1} = 2v - u_n$

To compute the Z-norm, which requires an accurate value of the expectation, a Monte-Carlo method is implemented with a 95% prediction interval and N_ω a maximum number of iterations. Algorithm 2 is used with $C_\alpha = 1.96$ to estimate this Z-norm.

5.2 The linear case

We mention the linear case because the exact solution is known and well defined at time t_n by its symbol

$$\widehat{u(t_n)}_k = e^{\frac{-ik}{1+k^2}(W_n - W_0)}\widehat{u}_{0k}.$$

Algorithm 2: Monte-Carlo method

Set u_0 and compute $\widehat{u_0}$.
Compute the solutions u^1, u^2 for two different Wiener processes using Algorithm 1.
Set $e^1 = \max_{1\le n\le N}(u^1(t_n)-u^1_n)$, $e^2 = \max_{1\le n\le N}(u^2(t_n)-u^2_n)$.
Compute the mean and the standard deviation
$\mu_2 := \frac{||e^1||^2_{H^1}+||e^2||^2_{H^1}}{2}$, $\sigma_2 := \sqrt{(||e^1||^2_{H^1}-\mu_2)^2+(||e^2||^2_{H^1}-\mu_2)^2}$.
while $2C_\alpha\sigma_j > \tau\mu_j\sqrt{j}$ and $j<N_\omega$ **do**
  Compute u^j,e^j for the j-th Wiener process
  Set $\mu_j := \frac{(j-1)\mu_{j-1}+||e^j||^2_{H^1}}{j}$, $\sigma_j := \left(\frac{(j-2)\sigma^2_{j-1}+(||e^j||^2_{H^1}-\mu_j)^2}{j-1}\right)^{1/2}$.
Set $||e^N||_Z := \sqrt{\mu_N}$.

For a pathway ω, the error $e^\omega_n = u(t_n)-u_n$ is immediately computable and its H^1-norm is provided by the Parseval equality $||e^\omega_n||^2_{H^1} = ||(1+k^2)\widehat{e^\omega_n}||^2_{L^2}$.

Simulations are made for $N_x = 2^{10}$ starting from the initial data $u_0(x) = 0.5e^{-x^2}$ until time $T=10$ for $\delta t = 2^m\times 5\times 10^{-6}$, $m=1,\cdots,6$, $N_\omega = 1000$, $\tau = 0.05$. Fig. 1 presents the error with respect to δt in a log-log scale. A linear regression shows a slope equals to 1.009. In Fig. 2, the space and time evolution of the approximate solution of the linear equation with $\delta t = 1.3\times 10^{-3}$ is represented for one Wiener process.

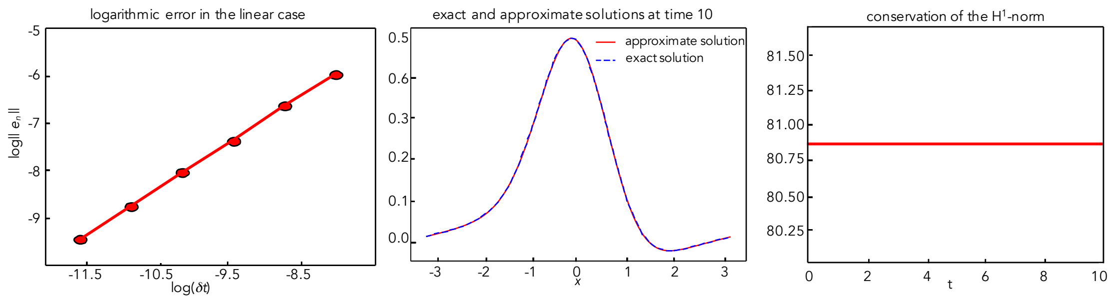

Figure 1: At left, order of convergence. At the center, exact solution and approximate solutions with $\delta t = 1.3\times 10^{-3}$ at time $T=10$. At right, conservation of the H^1 norm of a given pathwise solution with $\delta t = 1.3\times 10^{-3}$.

5.3 The nonlinear case

No exact stochastic solutions are known. To compute the error we consider a reference solution $u_{\delta t}$ with a time step $\delta t_{ref} = 5\times 10^{-6}$. This reference solution is compared to the approximations. Again, simulations are made with $N_x = 2^{10}$ starting from the initial data $u_0(x) = e^{-10x^2}$ until time $T = 10$ for $\delta t = 2^m\times 5\times 10^{-6}$, $m=1,\cdots,6$, $N_\omega = 1000$, $\tau = 0.05$,

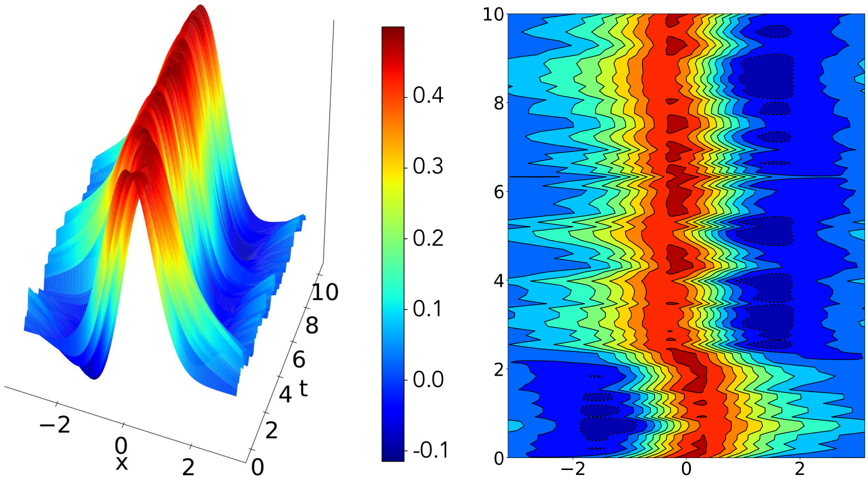

Figure 2: Space and time evolution of the approximate solution of the linear equation with $\delta t = 1.3\times 10^{-3}$.

$p=1$. Fig. 3 depicts the error with respect to δt in a log-log scale. A slope equals to 1.012 is obtained from a linear regression. Fig. 4 exhibits the space and time evolution of the approximate solution of the nonlinear equation with $p=1$, $\delta t = 1.3\times 10^{-3}$ and one Wiener process.

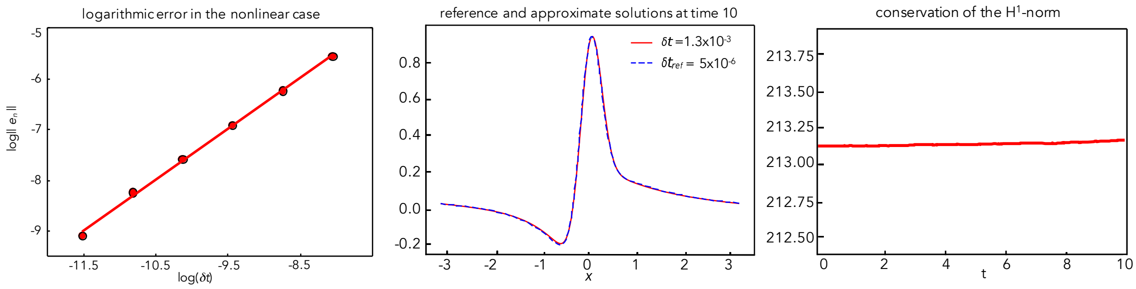

Figure 3: At left, order of convergence. At the center, reference solution and approximate solutions with $\delta t_{ref} = 5\times 10^{-6}$, and $\delta t = 1.3\times 10^{-3}$ respectively, at time $T=10$. At right, conservation of the H^1 norm of a given pathwise solution with $\delta t = 1.3\times 10^{-3}$.

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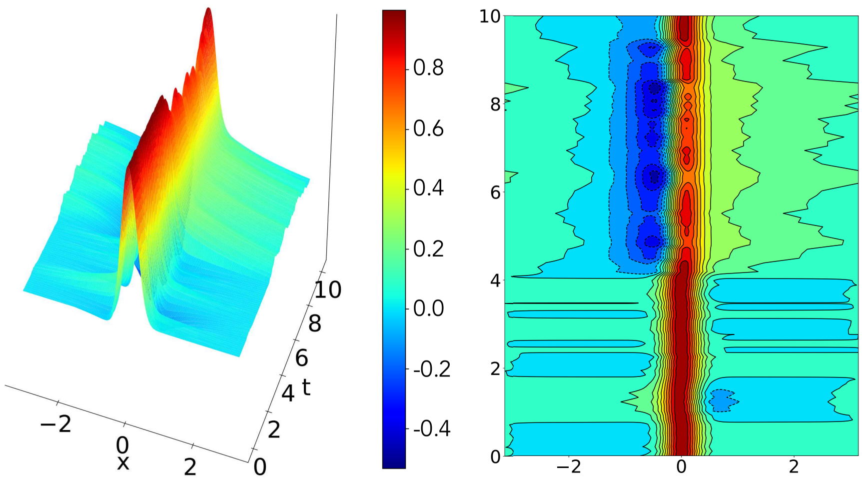

Figure 4: Space and time evolution of the approximate solution of the nonlinear equation with $p=1$ and $\delta t = 1.3\times 10^{-3}$.

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