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Finite Element Analysis for Nonstationary Magneto-Heat Coupling Problem

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Abstract. This paper is devoted to finite element analysis for the magneto-heat coupling model which governs the electromagnetic fields in large power transformers. The model, which couples Maxwell's equations and Heat equation through Ohmic heat source, is nonlinear. First we derive an equivalent weak formulation for the nonlinear magneto-heat model. We propose a linearized and temporally discrete scheme to approximate the continuous problem. The well-posedness and error estimates are proven for the semi-discrete scheme. Based on the results, we propose a fully discrete finite element problem and prove the error estimates for the approximate solutions. To validate the magneto-heat model and verify the efficiency of the finite element method, we compute an engineering benchmark problem of the International Compumag Society, P21^b-MN. The numerical results agree well with experimental data.

AMS subject classifications: 65N15, 65N30, 78A25

Key words: Magneto-heat coupling model, eddy current problem, Maxwell equations, finite element method.

1 Introduction

In numerical simulation of power transforms, eddy current loss accounts for the major part of the total energy loss. The Ohmic heat and magnetic hysteresis lead to energy loss and can damage the devices of a power transformer (see [8, 9]). In [18], the authors proposed a magnetic-heat coupling model for large power transformers and established

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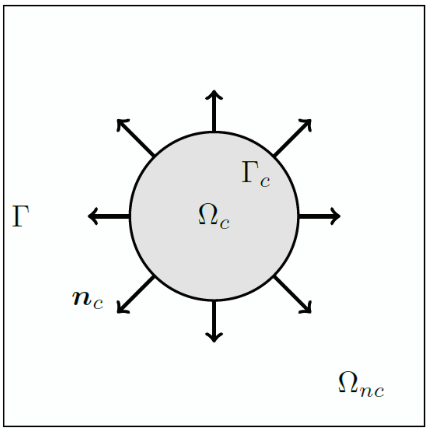

Figure 1: A 2D illustration of the problem geometry, $\bar{\Omega}=\bar{\Omega}_c\cup\bar{\Omega}_{nc}$.

the well-posedness of the problem. The model consists of Maxwell's equations and the heat equation with radiation condition on the boundary of conductors. It describes how the energy is transferred from electromagnetic fields to Ohmic heat, and the authors proposed a weak formulation for the model and established the well-posedness of the problem. This paper is a subsequent work of [18] and is focused on numerical analysis for the magneto-heat coupling model.

We study the magneto-heat coupling problem. Let $\Omega\subset\mathbb{R}^3$ be a truncation domain which contains all inhomogeneities like conductors, coils etc. The material outside of Ω is the air which is homogeneous and insulating. Without loss of generality, we assume Ω is a cube. Suppose $\bar{\Omega}=\bar{\Omega}_c\cup\bar{\Omega}_{nc}$ where Ω_c is the conducting region and Ω_{nc} the insulating region. Let $\Gamma=\partial\Omega$ denote the boundary of Ω and $\Gamma_c=\partial\Omega_c$ the boundary of Ω_c. Throughout the paper, we assume $\bar{\Omega}_c\subset\Omega$, $\Gamma\cap\Gamma_c=\emptyset$, and that Ω_c is a Lipschitz domain. Let T be the final time. The electromagnetic fields of the power transformer are governed by the Maxwell's equations

$$\partial_t\boldsymbol{B}+\mathbf{curl}\,\boldsymbol{E}=0 \quad \text{in } \Omega\times(0,T), \tag{1.1a}$$
$$\mathbf{curl}\,\boldsymbol{H}-\sigma\boldsymbol{E}=\boldsymbol{f} \quad \text{in } \Omega\times(0,T), \tag{1.1b}$$
$$\boldsymbol{E}\times\boldsymbol{n}=0 \quad \text{on } \Gamma\times(0,T), \tag{1.1c}$$
$$\boldsymbol{E}(0)=0 \quad \text{in } \Omega, \tag{1.1d}$$

where $\boldsymbol{E}$ stands for the electric field, $\boldsymbol{B}$ the magnetic flux density, $\boldsymbol{H}$ the magnetic field, σ the conductivity, and $\partial_t=\frac{\partial}{\partial t}$ the partial derivative with respect to t. The temperature θ is governed by the heat equation with initial and boundary conditions

$$C_\rho\partial_t\theta-\nabla\cdot(\kappa\nabla\theta)=\sigma|\boldsymbol{E}|^2 \quad \text{in } \Omega_c\times(0,T), \tag{1.2a}$$
$$-\kappa\partial_{\boldsymbol{n}_c}\theta=\bar{q}+\lambda(\theta,\partial_{\boldsymbol{n}_c}\theta)\cdot(\theta-\theta_0) \quad \text{on } \Gamma_c, \tag{1.2b}$$
$$\theta(0)=\theta_0 \quad \text{in } \Omega_c, \tag{1.2c}$$

where $\bar{q}$ is the heat flux through the boundary, θ_0 the background temperature, $C_\rho > 0$ the constant-volume specific heat times the material density, λ the radiation coefficient, and κ the thermal conductivity. Here $\boldsymbol{n}_c$ is the unit normal of Γ_c pointing to the exterior of Ω_c and $\partial_{\boldsymbol{n}_c}\theta = \frac{\partial\theta}{\partial \boldsymbol{n}_c}$. Moreover, the electric current density provides the Ohmic heat source $\sigma|\boldsymbol{E}|^2$ to (1.2a). It is in this way that problem (1.1) and problem (1.2) are coupled.

We remark that the boundary condition (1.2b) describes the heat emitted through the boundary of conducting materials. For example, a typical heat radiation condition reads as follows

$$-\kappa\partial_{\boldsymbol{n}_c}\theta = \bar{q} + h(\theta-\theta_0) + \epsilon(\theta^4-\theta_0^4) \quad \text{on } \Gamma_c,$$

where h is the heat transfer coefficient and ϵ the Stefan–Boltzmann constant (see [14]).

In [22], Preis et al. considered the case that the heat transfer coefficient and the conductivity depend only on steady temperature. They proposed one weak formulation of the eddy current problem by using the current vector potential and the magnetic scalar potential. The computational domain is restricted to conductors. In [21], Plasser et al. proposed a harmonic balance fixed-point approach to accelerate the convergence of the conjugate gradient method for magneto-heat coupling problem. The purpose of this paper is to establish rigorous error estimates for finite element solutions and to validate the magneto-heat coupling model by an engineering benchmark problem.

For linear eddy current problems, there are many interesting works in the literature on numerical methods (cf. e.g. [4, 6, 13, 16, 25]). For nonlinear eddy current problems, we refer to the recent works which are focused on developing efficient numerical methods. In [3], Bachinger et al. studied the numerical analysis of nonlinear multi-harmonic eddy current problems in isotropic materials. In [15], Jiang and Zheng proposed a new eddy current model for the Maxwell equations with laminated conductors based on the magnetic potential $\boldsymbol{A}$. The existence and uniqueness of the solution are proven for the new model. In [17], Li and Zheng proposed an approximate but effective $\boldsymbol{H}-\psi$ formulation for the nonlinear eddy current problem, which reduces the scale ratio by 2 orders of magnitude.

The layout of the paper is organized as follows. In Section 2 we present some notations and Sobolev spaces used in this paper and introduce some useful lemmas. In Section 3 we propose a reduced weak formulation for the magneto-heat coupling problem. In Section 4 we prove the well-posedness and error estimation of the discretization formulations. In Section 5 we present a numerical experiment to validate the theoretical results.

2 Preliminaries

The purpose of this section is to introduce some function spaces used in this paper and their associated norms and to fix the setting of the problem.

2.1 Function spaces

Let $D\subset\mathbb{R}^3$ be a bounded Lipschitz domain. The usual Hilbert space of square integrable functions on D is denoted by $L^2(D)$ which is equipped with the following inner product and norm

$$(u,v)_D:=\int_D u(\boldsymbol{x})v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \quad \text{and} \quad \|u\|_{L^2(D)}:=(u,v)_D^{1/2}.$$

For any nonnegative integer m, $H^m(D)$ denotes the subspace whose functions have square-integrable partial derivatives up to degree m. Let $H_0^1(D)$ be the subspace of $H^1(D)$ whose functions have zero traces on ∂D. Let $C^m([0,T])$ denote the space of all functions which have continuous derivatives on $[0,T]$ up to degree m. We denote vector-valued quantities by boldface notation, such as $\boldsymbol{L}^2(D):=(L^2(D))^3$.

For a Sobolev space X, we introduce the Sobolev–Bochner spaces

$$\begin{aligned}
&C^s(0,T;X):=\{u:[0,T]\to X:\ \|u\|_X\in C^s(0,T)\},\\
&L^2(0,T;X):=\left\{u:[0,T]\to X:\ \|u\|_{L^2(0,T;X)}<\infty\right\},\\
&H^1(0,T;X):=\{u\in L^2(0,T;X):\ \partial_t u\in L^2(0,T;X)\},
\end{aligned}$$

where

$$\|u\|^2_{L^2(0,T;X)}:=\int_0^T \|u\|^2_X,\ \|u\|^2_{H^1(0,T;X)}:=\|u\|^2_{L^2(0,T;X)}+\|\partial_t u\|^2_{L^2(0,T;X)}.$$

We define the spaces of functions having square integrable **curl** by

$$\begin{aligned}
&\boldsymbol{H}(\mathbf{curl},D):=\{\boldsymbol{v}\in\boldsymbol{L}^2(D):\ \mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{L}^2(D)\},\\
&\boldsymbol{H}_0(\mathbf{curl},D):=\{\boldsymbol{v}\in\boldsymbol{H}(\mathbf{curl},D):\ \boldsymbol{n}\times\boldsymbol{v}=0\ \text{ on } \partial D\},
\end{aligned}$$

which are equipped with the following inner product and norm

$$(\boldsymbol{v},\boldsymbol{w})_{\boldsymbol{H}(\mathbf{curl},D)}:=(\boldsymbol{v},\boldsymbol{w})_D+(\mathbf{curl}\,\boldsymbol{v},\mathbf{curl}\,\boldsymbol{w})_D,\quad \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},D))}:=\sqrt{(\boldsymbol{v},\boldsymbol{v})_{\boldsymbol{H}(\mathbf{curl},D)}}.$$

We shall also use the spaces of functions having square integrable divergence

$$\begin{aligned}
&\boldsymbol{H}(\mathrm{div},D):=\{\boldsymbol{v}\in\boldsymbol{L}^2(D):\ \mathrm{div}\,\boldsymbol{v}\in\boldsymbol{L}^2(D)\},\\
&\boldsymbol{H}_0(\mathrm{div},D):=\{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},D):\ \boldsymbol{n}\cdot\boldsymbol{v}=0\ \text{ on } \partial D\},
\end{aligned}$$

which are equipped with the following inner product and norm

$$(\boldsymbol{v},\boldsymbol{w})_{\boldsymbol{H}(\mathrm{div},D)}:=(\boldsymbol{v},\boldsymbol{w})_D+(\mathrm{div}\,\boldsymbol{v},\mathrm{div}\,\boldsymbol{w})_D,\quad \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div},D)}:=\sqrt{(\boldsymbol{v},\boldsymbol{v})_{\boldsymbol{H}(\mathrm{div},D)}}.$$

Define $\boldsymbol{X}_t(D)=\boldsymbol{H}_0(\mathbf{curl},D)\cap\boldsymbol{H}(\mathrm{div},D)$. Its norm is defined by

$$\|\boldsymbol{w}\|_{\boldsymbol{X}_t(D)}:=\left(\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2(D)}+\|\mathbf{curl}\,\boldsymbol{w}\|^2_{\boldsymbol{L}^2(D)}+\|\mathrm{div}\,\boldsymbol{w}\|^2_{L^2(D)}\right)^{1/2}.$$

From [11,19], $\boldsymbol{X}_t(D)$ is compactly embedded into $\boldsymbol{H}^s(D)$ for some $s>1/2$.

2.2 Space decomposition of $\boldsymbol{H}_0(\mathbf{curl},\Omega)$

To study the weak solution of (1.1) and (1.2), we shall also use the subspace

$$\boldsymbol{X}:=\{\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl},\Omega):\ (\boldsymbol{v},\nabla p)_\Omega=0\ \ \forall p\in H_c^1(\Omega)\},$$

where

$$H_c^1(\Omega):=\{p\in H_0^1(\Omega):\ p=\text{Constant}\ \text{ in }\bar{\Omega}_c\}.$$

Since $\bar{\Omega}_c\subset\Omega$ and $\Gamma\cap\Gamma_c=\emptyset$, the space $H_c^1(\Omega)$ is not empty. In fact, it contains any cutoff function $v\in C_0^\infty(\Omega)$ which satisfies $v\equiv1$ in Ω_c. Clearly $\boldsymbol{H}_0(\mathbf{curl},\Omega)$ admits the orthogonal decomposition

$$\boldsymbol{H}_0(\mathbf{curl},\Omega)=\boldsymbol{X}\oplus\nabla H_c^1(\Omega). \tag{2.1}$$

Lemma 2.1 ([3, Lemma 4]). *Let* $\boldsymbol{X}$ *be endowed with the inner product*

$$(\boldsymbol{v},\boldsymbol{w})_{\boldsymbol{X}}=\int_{\Omega_c}\boldsymbol{v}\cdot\boldsymbol{w}+\int_\Omega\mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{w}\quad\forall\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{X}. \tag{2.2}$$

Then $\|\cdot\|_{\boldsymbol{X}}=\sqrt{(\cdot,\cdot)_{\boldsymbol{X}}}$ *is an equivalent norm to* $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}$ *on* $\boldsymbol{X}$.

We also have the following property of the Sobolev–Bochner spaces $H^1(0,T;\boldsymbol{X})$ which will play an important role in our analysis.

Lemma 2.2 ([10, Section 5.9.2, Theorem 2]). *Let* $\boldsymbol{u}\in H^1(0,T;\boldsymbol{X})$. *Then* $\boldsymbol{u}\in C^0(0,T;\boldsymbol{X})$ *(after possibly being redefined on a set of measure zero) and there exists a constant C depending only on T such that*

$$\max_{0\le t\le T}\|\boldsymbol{u}(t)\|_{\boldsymbol{X}}\le C\|\boldsymbol{u}\|_{H^1(0,T;\boldsymbol{X})}. \tag{2.3}$$

Moreover, for any $0\le s\le t\le T$,

$$\boldsymbol{u}(t)=\boldsymbol{u}(s)+\int_s^t\partial_t\boldsymbol{u}(t')\,\mathrm{d}t'.$$

Lemma 2.3 ([15, Lemma 4.3]). *There exists a constant* $C_0>0$ *only depending only on* Ω *such that, for any* $\boldsymbol{f}\in\boldsymbol{L}^2(\Omega)$ *satisfying* $\operatorname{div}\boldsymbol{f}=0$,

$$|(\boldsymbol{f},\boldsymbol{v})_\Omega|\le C_0\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\quad\forall\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl},\Omega). \tag{2.4}$$

2.3 Problem setting

Throughout the paper, we make the following assumptions on the material parameters and the source current which are usually satisfied in electrical engineering. For details of material parameters, we refer to [7,8].

(H1) The electric conductivity σ is piecewise constant and there exist two constants $\sigma_{\min}$, $\sigma_{\max}$ satisfying

$$0<\sigma_{\min}\leq\sigma\leq\sigma_{\max} \quad \text{in } \Omega_c, \quad \sigma=0 \quad \text{in } \Omega_{nc}.$$

(H2) Let $\boldsymbol{H}=(H_1,H_2,H_3)$ and $\boldsymbol{B}=(B_1,B_2,B_3)$. Each H_i is a Lipschitz continuous function of B_i satisfying $B_i(0)=0$ and $H_i(B_i)=\mu_0^{-1}B_i$ in Ω_{nc} for $i=1,2,3$. Moreover, there exist two constants $\nu_{\min}$, $\nu_{\max}$ such that

$$0<\nu_{\min}\leq H_i'(B_i)\leq\nu_{\max} \quad \text{a.e. in } \Omega, \quad i=1,2,3,$$

where $\boldsymbol{\nu}:=\operatorname{diag}(H_1'(B_1),H_2'(B_2),H_3'(B_3))$ is usually defined by BH-curves. Fig. 2 shows the BH-curves in two different directions of the grain-oriented silicon steel laminations in large power transformers. Then the magnetic resistivity $\boldsymbol{\nu}=\boldsymbol{\nu}(\boldsymbol{B}):\boldsymbol{H}(\mathrm{div},\Omega)\to\mathbb{R}^{3\times 3}$ depends on the magnetic induction $\boldsymbol{B}$ and it is a Lipschitz continuous in $\boldsymbol{B}$.

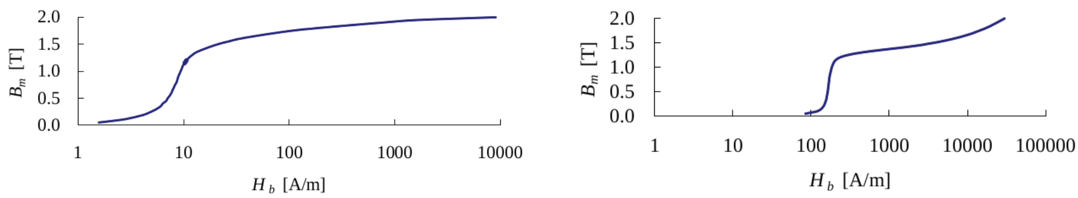

Figure 2: $\boldsymbol{BH}$-curves in rolling (left) and transverse (right) directions of steel.

(H3) The source current density $\boldsymbol{f}$ is carried in some coils and satisfies

$$\boldsymbol{f}\in C^2(0,T;\boldsymbol{L}^2(\Omega)), \quad \operatorname{supp}(\boldsymbol{f})\cap\bar{\Omega}_c=\emptyset, \quad \operatorname{div}\boldsymbol{f}=0.$$

(H4) The environment temperature θ_0 satisfies $\theta_0\in L^\infty(\bar{\Omega}_c)\cap H^1(\Omega_c)$.

(H5) Assume κ is piecewise constant and $\lambda=\lambda(\xi,\eta)$ is Lipschitz continuous in ξ,η. There are positive constants $\theta_{\min}$, $\lambda_{\min}$, $\lambda_{\max}$, $\kappa_{\min}$, and $\kappa_{\max}$ such that

$$\theta_{\min}\leq\theta_0, \qquad \lambda_{\min}<\lambda<\lambda_{\max}, \qquad \kappa_{\min}\leq\kappa\leq\kappa_{\max} \qquad \text{a.e. in } \Omega_c.$$

3 A reduced weak formulation

From (1.1a)-(1.1d), we easily get the following initial boundary value problem

$$\sigma\partial_t \boldsymbol{E}+\mathbf{curl}(\nu\,\mathbf{curl}\boldsymbol{E})=-\partial_t \boldsymbol{f} \quad \text{in } \Omega\times(0,T), \tag{3.1a}$$
$$\boldsymbol{E}\times\boldsymbol{n}=0 \quad \text{on } \Gamma\times(0,T), \tag{3.1b}$$
$$\boldsymbol{E}(0)=0 \quad \text{in } \Omega, \tag{3.1c}$$

where $\nu=\nu(\boldsymbol{B})$ is the magnetic resistivity. By Lemma 2.2 and Eq. (1.1a), the magnetic induction satisfies

$$\boldsymbol{B}(t)=-\int_0^t \mathbf{curl}\boldsymbol{E}(s)\mathrm{d}s.$$

A weak formulation equivalent to (3.1) reads: Find $\boldsymbol{E}\in \boldsymbol{L}^2(0,T;\boldsymbol{H}_0(\mathbf{curl},\Omega))$ such $\boldsymbol{E}(0)=0$ in Ω and

$$\int_\Omega \sigma\partial_t\boldsymbol{E}\cdot\boldsymbol{v}+\int_\Omega \nu\,\mathbf{curl}\boldsymbol{E}\cdot\mathbf{curl}\boldsymbol{v}=-\int_\Omega \partial_t\boldsymbol{f}\cdot\boldsymbol{v} \quad \forall \boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl},\Omega). \tag{3.2}$$

We remark that (3.2) is meant in the sense of distributions in time. It is obvious that the solution of (3.2) is not unique in the insulating region Ω_{nc}. Since $\mathrm{div}\,\boldsymbol{f}=0$, if $\boldsymbol{E}$ solves (3.2), then $\boldsymbol{E}+\nabla\phi$ also solve (3.2) for any $\phi\in H_c^1(\Omega)$.

From (2.1), let $\boldsymbol{E}$ be split into $\boldsymbol{E}=\boldsymbol{u}+\nabla\psi$ with $\boldsymbol{u}\in\boldsymbol{X}$ and $\psi\in H_c^1(\Omega)$. Then a reduced weak formulation is proposed on the subspace $\boldsymbol{X}$: Find $\boldsymbol{u}\in\boldsymbol{L}^2(0,T;\boldsymbol{X})$ such that $\boldsymbol{u}(0)=0$ in Ω and

$$\int_\Omega \sigma\partial_t\boldsymbol{u}\cdot\boldsymbol{v}+\int_\Omega \nu\,\mathbf{curl}\boldsymbol{u}\cdot\mathbf{curl}\boldsymbol{v}=-\int_\Omega \partial_t\boldsymbol{f}\cdot\boldsymbol{v} \quad \forall\boldsymbol{v}\in\boldsymbol{X}. \tag{3.3}$$

It is easy to see that $\boldsymbol{u}$ also satisfies (3.2). Although the solution $\boldsymbol{E}$ of (3.2) is not unique, the current density and the magnetic flux density are unique, namely,

$$\sigma\boldsymbol{E}=\sigma\boldsymbol{u}, \quad \boldsymbol{B}=-\int_0^t\mathbf{curl}\boldsymbol{E}=-\int_0^t\mathbf{curl}\boldsymbol{u} \quad \text{in } \Omega. \tag{3.4}$$

Therefore, we are only interested in $\sigma\boldsymbol{u}$ and $\mathbf{curl}\boldsymbol{u}$ throughout this paper.

To derive a weak formulation for the heat equation, we define

$$Y=H^1(\Omega_c).$$

Multiply both sides of (1.2a) with any $\varphi\in Y$. Using integration by part and the boundary conditions in (1.2b), we obtain

$$C_\rho\int_{\Omega_c}\partial_t\theta\varphi+\int_{\Omega_c}\kappa\nabla\theta\cdot\nabla\varphi+\int_{\Gamma_c}\lambda(\theta-\theta_0)\varphi=\int_{\Omega_c}\sigma|\boldsymbol{u}|^2\varphi. \tag{3.5}$$

Write $\theta=\eta+\theta_0$. From (3.3)-(3.5), we obtain a weak formulation of the magneto-heat equations: Find $\boldsymbol{u}\in \boldsymbol{L}^2(0,T;\boldsymbol{X})$ and $\eta\in L^2(0,T;Y)$ such that $\boldsymbol{u}(\cdot,0)=0$, $\eta(\cdot,0)=0$ and

$$\int_\Omega \sigma\partial_t\boldsymbol{u}\cdot\boldsymbol{v}+\int_\Omega \nu\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}=-\int_\Omega \partial_t\boldsymbol{f}\cdot\boldsymbol{v}, \tag{3.6a}$$

$$C_\rho\int_{\Omega_c}\partial_t\eta\varphi+a(\lambda;\eta,\varphi)=\int_{\Omega_c}\sigma|\boldsymbol{u}|^2\varphi-\int_{\Omega_c}\kappa\nabla\theta_0\cdot\nabla\varphi, \tag{3.6b}$$

for all $\boldsymbol{v}\in\boldsymbol{X}$ and $\varphi\in Y$, where

$$a(\lambda;\eta,\varphi):=\int_{\Omega_c}\kappa\nabla\eta\cdot\nabla\varphi+\int_{\Gamma_c}\lambda\eta\varphi. \tag{3.7}$$

The existence of the solutions to problem (3.6) can be proven by arguments similar to [18]. Here in this paper, we are only interested in finite element discretization of (3.6).

4 Finite element approximation

The purpose of this section is to propose the fully discrete scheme of (3.6) and to prove the a priori error estimates of discrete solutions.

4.1 The semi-discrete scheme

First we consider the semi-discrete scheme of (3.6) by discretizing the time variable. Let $0=t_0<t_1<\cdots<t_N=T$ be a uniform partition of $[0,T]$ and denote by $\tau=T/N$ the time step. Let $l^p(0,T;Z)$ denote the space of Z-valued sequences $w:=\{w_n:\, n=1,\cdots,N\}$ and define its norm $\|\cdot\|_{l^p(0,T;Z)}$ by

$$\|w\|_{l^p(0,T;Z)}:=\begin{cases}\left(\tau\sum_{n=1}^N\|w_n\|_Z^p\right)^{1/p} & \text{if } 1\le p<\infty,\\ \max\limits_{1\le n\le N}\|w_n\|_Z & \text{if } p=\infty.\end{cases}$$

A numerical scheme for solving (3.6) reads: Given $\boldsymbol{u}^0=0$ and $\eta^0=0$, find $(\boldsymbol{u}^n,\eta^n)\in \boldsymbol{X}\times Y$, $1\le n\le N$, such that

$$\int_\Omega \sigma\delta_t\boldsymbol{u}^n\cdot\boldsymbol{v}+\int_\Omega \nu^{n-1}\mathbf{curl}\,\boldsymbol{u}^n\cdot\mathbf{curl}\,\boldsymbol{v}=-\int_\Omega \delta_t\boldsymbol{f}^n\cdot\boldsymbol{v}, \tag{4.1a}$$

$$\int_{\Omega_c}C_\rho\delta_t\eta^n\varphi+a(\lambda^{n-1};\eta^n,\varphi)=\int_{\Omega_c}\sigma|\boldsymbol{u}^n|^2\varphi-\int_{\Omega_c}\kappa\nabla\theta_0\cdot\nabla\varphi, \tag{4.1b}$$

for any $(\boldsymbol{v},\varphi)\in\boldsymbol{X}\times Y$, where $\lambda^n=\lambda(\eta^n+\theta_0,\partial_{\boldsymbol{n}_c}\eta^n+\partial_{\boldsymbol{n}_c}\theta_0)$ and

$$\delta_t\boldsymbol{f}^n=\frac{\boldsymbol{f}(t_n)-\boldsymbol{f}(t_{n-1})}{\tau},\quad \delta_t\boldsymbol{u}^n=\frac{\boldsymbol{u}^n-\boldsymbol{u}^{n-1}}{\tau},\quad \delta_t\eta^n=\frac{\eta^n-\eta^{n-1}}{\tau}.$$

The magnetic resistivity is given by

$$\nu^n = \nu(\boldsymbol{B}^n), \quad \boldsymbol{B}^n = -\sum_{k=1}^{n} \tau \,\mathbf{curl}\, \boldsymbol{u}^k. \tag{4.2}$$

Lemma 4.1. *Let* (H1)-(H5) *be satisfied. Problem* (4.1) *has unique solutions for each* $1 \le n \le N$.

Proof. Rewrite (4.1) as: Find $(\boldsymbol{u}^n, \eta^n) \in \boldsymbol{X} \times Y$ such that

$$\int_\Omega \sigma \boldsymbol{u}^n \cdot \boldsymbol{v} + \tau \int_\Omega \nu^{n-1} \mathbf{curl}\, \boldsymbol{u}^n \cdot \mathbf{curl}\, \boldsymbol{v} = \int_\Omega \left(\sigma \boldsymbol{u}^{n-1} - \tau \delta_t \boldsymbol{f}^n \right) \cdot \boldsymbol{v}, \tag{4.3a}$$

$$C_\rho \int_{\Omega_c} \eta^n \varphi + \tau a(\lambda^{n-1}; \eta^n, \varphi) = \tau \int_{\Omega_c} \sigma |\boldsymbol{u}^n|^2 \varphi + C_\rho \int_{\Omega_c} \eta^{n-1} \varphi - \tau \int_{\Omega_c} \kappa \nabla \theta_0 \cdot \nabla \varphi. \tag{4.3b}$$

By Lemma 2.1 and assumptions (H1)-(H2), the left-hand side of (4.3a) provides a coercive and continuous bilinear form on $\boldsymbol{X} \times \boldsymbol{X}$. By Lemma 2.3, the right-hand side provides a linear and continuous functional on $\boldsymbol{X}$. We conclude that (4.3a) has a unique solution.

By (H5) and trace inequality, the left-hand side of (4.3b) provides a coercive and continuous bilinear form on $Y \times Y$. Moreover, taking $\boldsymbol{v} = \nabla \phi$ in (4.3a) with $\phi \in H^1(\Omega_c)$, we find that

$$\mathrm{div}\, \boldsymbol{u}^n = 0 \quad \text{in } \Omega_c, \quad \boldsymbol{u}^n \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega_c, \quad n = 0, 1, \cdots, N.$$

Therefore, $\boldsymbol{u}^n \in \boldsymbol{X}_t(\Omega_c) \hookrightarrow \boldsymbol{L}^q(\Omega_c)$ for some $q > 3$. From the Sobolev imbedding theorem $H^1(\Omega_c) \hookrightarrow L^6(\Omega_c)$, we infer that

$$\int_{\Omega_c} \sigma |\boldsymbol{u}^n|^2 \varphi \le C \|\boldsymbol{u}^n\|^2_{\boldsymbol{L}^3(\Omega_c)} \|\varphi\|_{L^6(\Omega_c)} \le C \|\boldsymbol{u}^n\|^2_{\boldsymbol{H}(\mathbf{curl}, \Omega_c)} \|\varphi\|_{H^1(\Omega_c)}. \tag{4.4}$$

The right-hand side of (4.3b) provides a linear and continuous functional on Y. This shows (4.3b) has a unique solution. □

Lemma 4.2. *Let* (H1)-(H5) *be satisfied and let* $\boldsymbol{u}^n, \eta^n$ *be the solutions of problem* (4.1) *for each* $1 \le n \le N$. *There exist two constants* $\tau_0 > 0$, $C > 0$ *which depend only on* T, Ω, $\|\boldsymbol{f}\|_{\boldsymbol{C}^1(0,T; \boldsymbol{L}^2(\Omega))}$, $\|\theta_0\|_{H^1(\Omega_c)}$, *and material parameters, such that, for any* $\tau \in (0, \tau_0]$,

$$\begin{aligned} &\|\boldsymbol{u}^{t_M}\|_{l^\infty(0,t_M; \boldsymbol{L}^2(\Omega))} + \|\mathbf{curl}\, \boldsymbol{u}^{t_M}\|_{l^2(0,t_M; \boldsymbol{L}^2(\Omega))} \le C, \\ &\|\eta^{t_M}\|_{l^\infty(0,t_M; L^2(\Omega_c))} + \|\nabla \eta^{t_M}\|_{l^2(0,t_M; \boldsymbol{L}^2(\Omega_c))} \le C, \end{aligned} \tag{4.5}$$

where $\boldsymbol{u}^{t_M} = \{\boldsymbol{u}^n : n = 1, \cdots, M\}$, $\eta^{t_M} = \{\eta^n : n = 1, \cdots, M\}$ *for* $M = 1, 2, \cdots, N$.

Proof. Setting $\boldsymbol{v} = \boldsymbol{u}^n$ in (4.3a) shows that

$$\int_\Omega \sigma (\boldsymbol{u}^n - \boldsymbol{u}^{n-1}) \cdot \boldsymbol{u}^n + \tau \int_\Omega \nu^{n-1} \mathbf{curl}\, \boldsymbol{u}^n \cdot \mathbf{curl}\, \boldsymbol{u}^n = -\tau \int_\Omega \delta_t \boldsymbol{f}^n \cdot \boldsymbol{u}^n. \tag{4.6}$$

Using the initial value $\boldsymbol{u}^0=0$ and the inequality

$$2\int_\Omega \sigma(\boldsymbol{u}^n-\boldsymbol{u}^{n-1})\cdot\boldsymbol{u}^n \geq \sigma_{\min}\left(\|\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega_c)}-\left\|\boldsymbol{u}^{n-1}\right\|^2_{\boldsymbol{L}^2(\Omega_c)}\right),$$

we have

$$2\sum_{n=1}^m\int_\Omega \sigma(\boldsymbol{u}^n-\boldsymbol{u}^{n-1})\cdot\boldsymbol{u}^n \geq \sigma_{\min}\|\boldsymbol{u}^m\|^2_{\boldsymbol{L}^2(\Omega_c)}. \tag{4.7}$$

Inserting (4.7) into (4.6) and summing up (4.6) for $1\leq n\leq m$, we have

$$\frac{1}{2}\sigma_{\min}\|\boldsymbol{u}^m\|^2_{\boldsymbol{L}^2(\Omega)}+\nu_{\min}\sum_{n=1}^m\tau\|\mathbf{curl}\,\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}\leq\tau\sum_{n=1}^m\left|\int_\Omega\delta_t\boldsymbol{f}^n\cdot\boldsymbol{u}^n\right|. \tag{4.8}$$

Assumption (H3) implies $\operatorname{div}(\delta_t\boldsymbol{f}^n)=0$. Using Lemma 2.3, we have

$$\left|\int_\Omega\delta_t\boldsymbol{f}^n\cdot\boldsymbol{u}^n\right|\leq\frac{C_0}{\nu_{\min}}\|\delta_t\boldsymbol{f}^n\|^2_{\boldsymbol{L}^2(\Omega)}+\frac{\nu_{\min}}{2}\|\mathbf{curl}\,\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}, \tag{4.9}$$

where C_0 depending only on Ω. Inserting (4.9) into (4.8) and writing $\delta_t\boldsymbol{f}=\{\delta_t\boldsymbol{f}^n:n=1,\cdots,N\}$, we have

$$\sigma_{\min}\|\boldsymbol{u}^m\|^2_{\boldsymbol{L}^2(\Omega_c)}+\nu_{\min}\sum_{n=1}^m\tau\|\mathbf{curl}\,\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}\leq 2TC_0\nu_{\min}^{-1}\|\delta_t\boldsymbol{f}\|^2_{l^2(0,T;\boldsymbol{L}^2(\Omega))}.$$

It follows that

$$\max_{1\leq n\leq N}\|\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega_c)}+\sum_{n=1}^m\tau\|\mathbf{curl}\,\boldsymbol{u}^n\|^2_{\boldsymbol{L}^2(\Omega)}\leq C\|\boldsymbol{f}\|^2_{\boldsymbol{C}^1(0,T;\boldsymbol{L}^2(\Omega))}. \tag{4.10}$$

Setting $\varphi=\eta^n$ in (4.3b) and noting $\int_{\Gamma_c}\lambda^{n-1}|\eta^n|^2\geq 0$, we find that

$$C_\rho\int_{\Omega_c}(\eta^n-\eta^{n-1})\eta^n+\tau\int_{\Omega_c}\kappa|\nabla\eta^n|^2\leq\tau\int_{\Omega_c}\left(\sigma|\boldsymbol{u}^n|^2\eta^n-\kappa\nabla\theta_0\cdot\nabla\eta^n\right). \tag{4.11}$$

Similarly, the initial value $\eta^0=0$ implies

$$2C_\rho\int_{\Omega_c}(\eta^n-\eta^{n-1})\eta^n\geq C_\rho\left(\|\eta^n\|^2_{L^2(\Omega_c)}-\left\|\eta^{n-1}\right\|^2_{L^2(\Omega_c)}\right). \tag{4.12}$$

By the proof of Lemma 4.1, we have $\boldsymbol{u}^n\in\boldsymbol{X}_t(\Omega_c)\hookrightarrow\boldsymbol{L}^q(\Omega_c)$ for $q>3$. From (4.10) and the Sobolev imbedding theorem $H^1(\Omega_c)\hookrightarrow L^6(\Omega_c)$, we have

$$\begin{aligned}\left|\int_{\Omega_c}\sigma|\boldsymbol{u}^n|^2\eta^n\right|&\leq C\|\boldsymbol{u}^n\|_{\boldsymbol{L}^2(\Omega_c)}\|\boldsymbol{u}^n\|_{\boldsymbol{L}^3(\Omega_c)}\|\eta^n\|_{L^6(\Omega_c)}\leq C\|\boldsymbol{u}^n\|_{\boldsymbol{H}(\mathbf{curl},\Omega_c)}\|\eta^n\|_{H^1(\Omega_c)}\\&\leq C\|\boldsymbol{u}^n\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega_c)}+C\|\eta^n\|^2_{L^2(\Omega_c)}+\frac{1}{4}\left\|\kappa^{1/2}\nabla\eta^n\right\|^2_{\boldsymbol{L}^2(\Omega_c)},\end{aligned} \tag{4.13}$$

$$\left|\int_{\Omega_c}\kappa\nabla\theta_0\cdot\nabla\eta^n\right|\leq C\|\nabla\theta_0\|^2_{\boldsymbol{L}^2(\Omega_c)}+\frac{1}{4}\left\|\kappa^{1/2}\nabla\eta^n\right\|^2_{\boldsymbol{L}^2(\Omega_c)}. \tag{4.14}$$

Now inserting (4.12)-(4.14) into (4.11) and summing up the result for $1\le n\le m$, we get

$$C_\rho\|\eta^m\|^2_{L^2(\Omega_c)}+\frac{1}{2}\sum_{n=1}^m\tau\left\|\kappa^{1/2}\nabla\eta^n\right\|^2_{\boldsymbol{L}^2(\Omega_c)}\le C+C\sum_{n=1}^m\tau\|\eta^n\|^2_{L^2(\Omega_c)}.$$

By discrete Gronwall inequality's in [12], we have

$$\|\eta^m\|^2_{L^2(\Omega_c)}+\sum_{n=1}^m\tau\|\nabla\eta^n\|^2_{\boldsymbol{L}^2(\Omega_c)}\le C. \tag{4.15}$$

This completes the proof. □

Now we estimate the errors between the approximate solutions and the exact solutions. Define

$$\boldsymbol{e}^n_{\boldsymbol{u}}=\boldsymbol{u}^n-\boldsymbol{u}(t_n),\quad e^n_\eta=\eta^n-\eta(t_n),\quad n=1,\cdots,N.$$

Lemma 4.3. *Let* (H1)-(H5) *be satisfied and assume that, for some* $p>3$ *and* $q>2$,

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{C}^1(0,T;\boldsymbol{L}^2(\Omega))}+\|\mathbf{curl}\,\boldsymbol{u}\|_{H^1(0,T;\boldsymbol{W}^{1,p}(\Omega))}+\|\eta\|_{W^{1+1/q,q}(\Omega_c)}\le C, \tag{4.16}$$

where the constant $C>0$ *is independent of* τ. *Then*

$$\int_\Omega\left|(\boldsymbol{\nu}-\boldsymbol{\nu}^{n-1})\mathbf{curl}\,\boldsymbol{u}(t_n)\cdot\mathbf{curl}\,\boldsymbol{e}^n_{\boldsymbol{u}}\right|\le C\tau\left(1+\sum_{k=0}^{n-1}\left\|\mathbf{curl}\,\boldsymbol{e}^k_{\boldsymbol{u}}\right\|_{\boldsymbol{L}^2(\Omega)}\right)\|\mathbf{curl}\,\boldsymbol{e}^n_{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}, \tag{4.17}$$

$$\int_{\Gamma_c}\left|(\lambda-\lambda^{n-1})\eta(t_n)e^n_\eta\right|\le C\sum_{k=n-1}^{n}\left\|e^k_\eta\right\|_{L^2(\Omega_c)}\left\|e^k_\eta\right\|_{H^1(\Omega_c)}. \tag{4.18}$$

Proof. Recall the injection $\boldsymbol{W}^{1,p}(\Omega)\hookrightarrow\boldsymbol{C}_B(\Omega)$ for $p>3$ where $\boldsymbol{C}_B(\Omega)$ is the space of continuous and bounded functions on Ω. By Lemma 2.2, we have

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^\infty((0,T);\boldsymbol{L}^\infty(\Omega))}\le\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^\infty((0,T);\boldsymbol{W}^{1,p}(\Omega))}\le\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{H}^1((0,T);\boldsymbol{W}^{1,p}(\Omega))}\le C.$$

So using the Lipschitz continuous of $\boldsymbol{\nu}$, we have

$$\left|\int_\Omega(\boldsymbol{\nu}-\boldsymbol{\nu}^{n-1})\mathbf{curl}\,\boldsymbol{u}(t_n)\cdot\mathbf{curl}\,\boldsymbol{e}^n_{\boldsymbol{u}}\right|\le C\left\|\boldsymbol{B}(t_n)-\boldsymbol{B}^{n-1}\right\|_{\boldsymbol{L}^2(\Omega)}\|\mathbf{curl}\,\boldsymbol{e}^n_{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}. \tag{4.19}$$

From (3.4) and (4.2), we find that

$$\begin{aligned}\left|\boldsymbol{B}(t_n)-\boldsymbol{B}^{n-1}\right|&=\left|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(\mathbf{curl}\,\boldsymbol{u}-\mathbf{curl}\,\boldsymbol{u}^k)\right|\\&\le\sum_{k=0}^{n-1}\left|\int_{t_k}^{t_{k+1}}(\mathbf{curl}\,\boldsymbol{u}-\mathbf{curl}\,\boldsymbol{u}(t_k))\right|+\tau\sum_{k=0}^{n-1}\left|\mathbf{curl}\,\boldsymbol{e}^k_{\boldsymbol{u}}\right|\\&\le\tau\int_0^T|\partial_t\mathbf{curl}\,\boldsymbol{u}|+\tau\sum_{k=0}^{n-1}\left|\mathbf{curl}\,\boldsymbol{e}^k_{\boldsymbol{u}}\right|\le C\tau+\tau\sum_{k=0}^{n-1}\left|\mathbf{curl}\,\boldsymbol{e}^k_{\boldsymbol{u}}\right|.\end{aligned} \tag{4.20}$$

Inserting (4.20) into (4.19) yields (4.17).

By the regularity assumptions (4.16) and Sobolev imbedding theorems, we have

$$H^{-1/2}(\Gamma_c) \hookrightarrow L^{4/3}(\Gamma_c), \quad H^{1/2}(\Gamma_c) \hookrightarrow L^4(\Gamma_c), \quad W^{1,q}(\Gamma_c) \hookrightarrow L^\infty(\Gamma_c), \quad q>2.$$

Since λ is Lipschitz continuous, using the trace theorem, we have

$$\int_{\Gamma_c} \left|(\lambda-\lambda^{n-1})\eta(t_n)e_\eta^n\right| \le C\sum_{k=n-1}^{n} \left\|e_\eta^k\right\|^2_{L^2(\Gamma_c)} \le C\sum_{k=n-1}^{n} \left\|e_\eta^k\right\|_{L^2(\Omega_c)} \left\|e_\eta^k\right\|_{H^1(\Omega_c)}.$$

The proof is completed. □

Theorem 4.1. *Let* (H1)-(H5) *be satisfied and assume the solutions of* (3.6) *satisfy*

$$\|\boldsymbol{u}\|_{\boldsymbol{C}^2(0,T;\boldsymbol{X})} + \|\eta\|_{C^2(0,T;Y)} \le C. \tag{4.21}$$

Then for sufficiently small τ, it holds

$$\left\|\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\right\|_{l^\infty(0,t_M;\boldsymbol{L}^2(\Omega))} + \left\|\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\right\|_{l^2(0,t_M;\boldsymbol{L}^2(\Omega))} \le C\tau, \tag{4.22a}$$

$$\left\|e_\eta^{t_M}\right\|_{l^\infty(0,t_M;L^2(\Omega_c))} + \left\|\nabla e_\eta^{t_M}\right\|_{l^2(0,t_M;L^2(\Omega_c))} \le C\tau, \tag{4.22b}$$

where $\boldsymbol{e}_{\boldsymbol{u}}^{t_M} = \{\boldsymbol{e}_{\boldsymbol{u}}^1,\cdots,\boldsymbol{e}_{\boldsymbol{u}}^M\}$, $e_\eta^{t_M} = \{e_\eta^1,\cdots,e_\eta^M\}$ *for* $M=1,2,\cdots,N$.

Proof. Using (3.6) and (4.1), there exist $\zeta_n,\xi_n,\gamma_n \in (t_{n-1},t_n)$ such that

$$\begin{aligned}
&\int_\Omega \sigma(\boldsymbol{e}_{\boldsymbol{u}}^n - \boldsymbol{e}_{\boldsymbol{u}}^{n-1})\cdot\boldsymbol{v} + \tau\int_\Omega \nu^{n-1}\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^n\cdot\mathbf{curl}\,\boldsymbol{v} \\
=&\tau\int_\Omega (\nu-\nu^{n-1})\mathbf{curl}\,\boldsymbol{u}(t_n)\cdot\mathbf{curl}\,\boldsymbol{v} + \frac{\tau^2}{2}\int_\Omega [\partial_{tt}\boldsymbol{f}(\zeta_n) + \sigma\partial_{tt}\boldsymbol{u}(\xi_n)]\cdot\boldsymbol{v} \qquad \forall \boldsymbol{v}\in\boldsymbol{X}, \\
&\int_{\Omega_c} C_\rho(e_\eta^n - e_\eta^{n-1})\varphi + \tau a(\lambda^{n-1},e_\eta^n,\varphi) + \tau\int_{\Omega_c}\sigma\left[\boldsymbol{u}(t_n)^2 - |\boldsymbol{u}^n|^2\right]\varphi \\
=&\tau\int_{\Gamma_c}(\lambda-\lambda^{n-1})\eta(t_n)\varphi + \frac{\tau^2}{2}C_\rho\int_{\Omega_c}\partial_{tt}\eta(\gamma_n)\varphi \qquad \forall\varphi\in Y.
\end{aligned}$$

Taking $(\boldsymbol{v},\varphi) = (\boldsymbol{e}_{\boldsymbol{u}}^n, e_\eta^n)$ and using $\int_{\Gamma_c}\lambda^{n-1}|e_\eta^n|^2 \ge 0$, we have

$$\int_\Omega \sigma(\boldsymbol{e}_{\boldsymbol{u}}^n - \boldsymbol{e}_{\boldsymbol{u}}^{n-1})\cdot\boldsymbol{e}_{\boldsymbol{u}}^n + \tau\int_\Omega \nu^{n-1}|\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^n|^2 = \tau\sum_{j=1}^{2} I_j^{(n)}, \tag{4.23a}$$

$$\int_{\Omega_c} C_\rho(e_\eta^n - e_\eta^{n-1})e_\eta^n + \tau\int_{\Omega_c}\kappa|\nabla e_\eta^n|^2 \le \tau\sum_{j=1}^{3} J_j^{(n)}. \tag{4.23b}$$

where

$$I_1^{(n)}:=\int_\Omega(\boldsymbol{\nu}-\boldsymbol{\nu}^{n-1})\,\mathbf{curl}\,\boldsymbol{u}(t_n)\cdot\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^n,\ I_2^{(n)}:=\frac{\tau}{2}\int_\Omega[\partial_{tt}\boldsymbol{f}(\zeta_n)+\sigma\partial_{tt}\boldsymbol{u}(\xi_n)]\cdot\boldsymbol{e}_{\boldsymbol{u}}^n,$$
$$J_1^{(n)}:=\int_{\Gamma_c}(\lambda-\lambda^n)\eta(t_n)e_\eta^n,\quad J_2^{(n)}:=\frac{\tau}{2}\int_{\Omega_c}C_\rho\partial_{tt}\eta(\gamma_n)e_\eta^n,\quad J_3^{(n)}:=\int_{\Omega_c}\sigma\boldsymbol{e}_{\boldsymbol{u}}^n\cdot[\boldsymbol{e}_{\boldsymbol{u}}^n+2\boldsymbol{u}(t_n)]e_\eta^n.$$

Recall from Lemma 4.3 that

$$\begin{aligned}\sum_{n=1}^M I_1^{(n)}&\le C\tau\sum_{n=1}^M\left(1+\sum_{k=0}^{n-1}\left\|\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^k\right\|_{\boldsymbol{L}^2(\Omega)}\right)\|\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega)}\\&\le C\left\|\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\right\|_{l^2(0,t_M;\boldsymbol{L}^2(\Omega))}+C\left\|\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\right\|^2_{l^2(0,t_M;\boldsymbol{L}^2(\Omega))},\\ \sum_{n=1}^M\tau J_1^{(n)}&\le C\sum_{n=1}^M\tau\left\|e_\eta^n\right\|^2_{L^2(\Omega_c)}+\frac{1}{2}\kappa_{\min}\sum_{n=1}^M\tau\left\|\nabla e_\eta^n\right\|^2_{\boldsymbol{L}^2(\Omega_c)}\\&=C\left\|e_\eta^{t_M}\right\|_{l^2(0,t_M;L^2(\Omega_c))}+\frac{1}{2}\kappa_{\min}\left\|\nabla e_\eta^{t_M}\right\|_{l^2(0,t_M;\boldsymbol{L}^2(\Omega_c))}.\end{aligned}$$

Moreover, $I_2^{(n)}$ and $J_2^{(n)}$ can be estimated as follows

$$\begin{aligned}&\sum_{n=1}^M I_2^{(n)}\le C\left(\|\boldsymbol{f}\|_{\boldsymbol{H}^2(0,t_M;\boldsymbol{L}^2(\Omega))}+\|\boldsymbol{u}\|_{\boldsymbol{H}^2(0,t_M;\boldsymbol{L}^2(\Omega))}\right)\|\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\|_{l^2(0,t_M;\boldsymbol{L}^2(\Omega))}\le C\|\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\|_{l^2(0,t_M;\boldsymbol{L}^2(\Omega))},\\&\sum_{n=1}^M\tau J_2^{(n)}\le C\tau\|\eta\|_{\boldsymbol{H}^2(0,t_M;L^2(\Omega_c))}\left\|e_\eta^{t_M}\right\|_{l^2(0,t_M;L^2(\Omega))}\le\tau^2+C\left\|e_\eta^{t_M}\right\|^2_{l^2(0,t_M;L^2(\Omega))}.\end{aligned}$$

Next we estimate $J_3^{(n)}$. Using Lemma 4.2 and arguments similar to (4.4), we know that

$$\begin{aligned}\int_{\Omega_c}\sigma|\boldsymbol{e}_{\boldsymbol{u}}^n|^2e_\eta^n&\le\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega_c)}\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^3(\Omega_c)}\left\|e_\eta^n\right\|_{L^6(\Omega_c)}\le C\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_{\boldsymbol{X}_t(\Omega_c)}\left\|e_\eta^n\right\|_{H^1(\Omega_c)}\\&\le C\|\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^n\|_{\boldsymbol{L}^2(\Omega_c)}\left\|e_\eta^n\right\|_{H^1(\Omega_c)}.\end{aligned}$$

This shows

$$\sum_{n=1}^M\tau J_3^{(n)}\le C\|\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\|_{l^2(0,t_M;\boldsymbol{H}(\mathbf{curl},\Omega_c))}\left\|e_\eta^{t_M}\right\|_{l^2(0,t_M;L^2(\Omega))}.$$

Substituting the estimates for $I_1^{(n)}$ and $I_2^{(n)}$ into (4.23a) shows that

$$\begin{aligned}&\sum_{n=1}^M\int_\Omega\sigma(\boldsymbol{e}_{\boldsymbol{u}}^n-\boldsymbol{e}_{\boldsymbol{u}}^{n-1})\cdot\boldsymbol{e}_{\boldsymbol{u}}^n+\nu_{\min}\|\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\|^2_{l^2(0,t_M;\boldsymbol{L}^2(\Omega))}\\&\le C\tau\left(\|\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\|_{l^2(0,t_M;\boldsymbol{L}^2(\Omega))}+\|\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\|_{l^2(0,t_M;\boldsymbol{L}^2(\Omega))}+\|\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\|^2_{l^2(0,t_M;\boldsymbol{L}^2(\Omega))}\right).\end{aligned}$$

Note that $\boldsymbol{e}_{\boldsymbol{u}}^0=0$ and

$$\int_\Omega \sigma(\boldsymbol{e}_{\boldsymbol{u}}^n-\boldsymbol{e}_{\boldsymbol{u}}^{n-1})\cdot\boldsymbol{e}_{\boldsymbol{u}}^n \geq \left\|\sigma^{1/2}\boldsymbol{e}_{\boldsymbol{u}}^n\right\|^2_{\boldsymbol{L}^2(\Omega)}-\left\|\sigma^{1/2}\boldsymbol{e}_{\boldsymbol{u}}^{n-1}\right\|^2_{\boldsymbol{L}^2(\Omega)}.$$

Summing the above inequalities for $n=1,\cdots,M$ and assuming τ small enough, we have

$$\begin{aligned}\left\|\boldsymbol{e}_{\boldsymbol{u}}^M\right\|^2_{\boldsymbol{L}^2(\Omega)}+\left\|\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\right\|^2_{l^2(0,t_M;\boldsymbol{L}^2(\Omega))} &\leq C\tau\left(\left\|\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\right\|_{l^2(0,t_M;\boldsymbol{L}^2(\Omega))}+\left\|\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\right\|_{l^2(0,t_M;\boldsymbol{L}^2(\Omega))}\right)\\ &\leq C\tau^2+\left\|\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\right\|^2_{l^2(0,t_M;\boldsymbol{L}^2(\Omega))}+\frac{1}{2}\left\|\mathbf{curl}\,\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\right\|^2_{l^2(0,t_M;\boldsymbol{L}^2(\Omega))}.\end{aligned}$$

Then we obtain (4.22a) by using Gronwall's inequality.

Similarly, since $e_\eta=0$ and

$$2\int_{\Omega_c}(e_\eta^n-e_\eta^{n-1})e_\eta^n \geq \left\|e_\eta^n\right\|^2_{L^2(\Omega_c)}-\left\|e_\eta^{n-1}\right\|^2_{L^2(\Omega_c)},$$

substituting the estimates for $J_1^{(n)},J_2^{(n)},J_3^{(n)}$ into (4.23b), we have

$$\begin{aligned}C_\rho\left\|e_\eta^M\right\|^2_{L^2(\Omega_c)}+\sum_{n=1}^M\tau\int_{\Omega_c}\kappa|\nabla e_\eta^n|^2 &\leq \sum_{n=1}^M\left[C_\rho\int_{\Omega_c}(e_\eta^n-e_\eta^{n-1})e_\eta^n+\tau\int_{\Omega_c}\kappa|\nabla e_\eta^n|^2\right]\\ &\leq C\left\|e_\eta^{t_M}\right\|^2_{l^2(0,t_M;L^2(\Omega))}+\frac{1}{2}\kappa_{\min}\sum_{n=1}^M\tau\left\|\nabla e_\eta^n\right\|^2_{\boldsymbol{L}^2(\Omega_c)}+\tau^2\\ &\quad +C\left\|\boldsymbol{e}_{\boldsymbol{u}}^{t_M}\right\|_{l^2(0,t_M;\boldsymbol{H}(\mathbf{curl},\Omega_c))}\left\|e_\eta^{t_M}\right\|_{l^2(0,t_M;L^2(\Omega))}.\end{aligned}$$

We get (4.22b) by using (4.22a) and Gronwall's inequality. □

4.2 Fully discrete scheme

Let $\mathcal{T}_h$ be a quasi-uniform and shape-regular tetrahedral triangulation of Ω such that $\mathcal{T}_h^c:=\mathcal{T}_h|_{\Omega_c}$ provides a tetrahedral triangulation of Ω_c. Let h_K be the diameter of an element $K\in\mathcal{T}_h$ and define $h=\max_{K\in\mathcal{T}_h}h_K\leq 1$. Let P_k be the space of all polynomials with degree less than or equal to k. First we introduce the l^{th}-order ($l\leq k$) Lagrange finite element spaces on Ω and Ω_c respectively by

$$V_h=\left\{v\in H^1(\Omega):\ v|_K\in P_l(K)\ \ \forall K\in\mathcal{T}_h^c\right\},\quad Y_h=\left\{v\in H^1(\Omega_c): v|_K\in P_l(K)\ \ \forall K\in\mathcal{T}_h^c\right\}.$$

The k^{th}-order Nédélec edge element spaces in the second family are defined by

$$\boldsymbol{V}_h=\{\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl},\Omega):\boldsymbol{v}|_K\in\boldsymbol{P}_k(K),\ \forall K\in\mathcal{T}_h\},\quad \boldsymbol{X}_h=\{\boldsymbol{v}\in\boldsymbol{V}_h:\ (\boldsymbol{v},\nabla\phi)_{\Omega_c}=0\ \ \forall\phi\in V_h^c\},$$

where $V_h^c=\{v\in V_h:\ \ v=\text{Const in }\bar{\Omega}_c\}$.

The fully discrete problem to problem (3.6) reads: Given $\boldsymbol{u}_h^0=0$, $\eta_h^0=0$, find $(\boldsymbol{u}_h^n,\eta_h^n)\in \boldsymbol{X}_h\times Y_h$ such that

$$\int_\Omega \sigma\delta_t \boldsymbol{u}_h^n\cdot \boldsymbol{v}_h+\int_\Omega \boldsymbol{\nu}_h^{n-1}\mathbf{curl}\,\boldsymbol{u}_h^n\cdot \mathbf{curl}\,\boldsymbol{v}_h=-\int_\Omega \delta_t \boldsymbol{f}^n\cdot \boldsymbol{v}_h, \tag{4.24a}$$

$$\int_{\Omega_c} C_\rho\delta_t\eta_h^n\varphi_h+a(\lambda_h^{n-1},\eta_h^n,\varphi_h)=\int_{\Omega_c}\sigma|\boldsymbol{u}_h^n|^2\varphi_h-\int_{\Omega_c}\kappa\nabla\theta_0\cdot\nabla\varphi_h, \tag{4.24b}$$

for any $(\boldsymbol{v}_h,\varphi_h)\in \boldsymbol{X}_h\times Y_h$, where $\lambda_h^n:=\lambda(\eta_h^n+\theta_0,\partial_{\boldsymbol{n}_c}\eta_h^n+\partial_{\boldsymbol{n}_c}\theta_0)$ and the magnetic resistivity is given by

$$\boldsymbol{\nu}_h^n=\boldsymbol{\nu}_h^n(\boldsymbol{B}_h^n),\ \boldsymbol{B}_h^n=-\sum_{k=1}^n \tau\,\mathbf{curl}\,\boldsymbol{u}_h^k. \tag{4.25}$$

The existence and uniqueness for (4.24) can be proven by arguments similar to the proof of Lemma 4.2. From [5,19,20,23], we have the following estimates:

- for any $1\le p\le\infty$, $1\le q\le\infty$, and $0<m\le l$, there is a constant $C\equiv C(l,p,q,\rho)$ such that

$$\left[\sum_{K\in\mathcal{T}_h}\|v\|_{W^{l,p}(K)}^p\right]^{1/p}\le Ch^{m-l+\min(0,\frac{d}{p}-\frac{d}{q})}\left[\sum_{K\in\mathcal{T}_h}\|v\|_{W^{m,q}(K)}^q\right]^{1/q}\quad \forall v\in Y_h; \tag{4.26}$$

- there exists an interpolation operator $\Pi_h:\boldsymbol{X}\to\boldsymbol{X}_h$ such that

$$\|\boldsymbol{v}-\Pi_h\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\le Ch^k\|\boldsymbol{v}\|_{\boldsymbol{H}^k(\Omega)}, \tag{4.27a}$$

$$\|\mathbf{curl}(\boldsymbol{v}-\Pi_h\boldsymbol{v})\|_{\boldsymbol{L}^2(\Omega)}\le Ch^{k-1}\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{H}^{k-1}(\Omega)}; \tag{4.27b}$$

- and there exists an interpolation operator $R_h:Y\to Y_h$ such that

$$\|\varphi-R_h\varphi\|_{L^2(\Omega_c)}+h\|\nabla(\varphi-R_h\varphi)\|_{L^2(\Omega_c)}\le Ch^l\|\varphi\|_{H^l(\Omega_c)}. \tag{4.28}$$

Our purpose is ti estimate the errors $\boldsymbol{F}_{\boldsymbol{u}}^n:=\boldsymbol{u}_h^n-\boldsymbol{u}(t_n)$ and $F_\eta^n:=\eta_h^n-\eta(t_n)$. Using the decompositions

$$\begin{aligned}\boldsymbol{F}_{\boldsymbol{u}}^n&=[\boldsymbol{u}_h^n-\Pi_h\boldsymbol{u}^n]+[\Pi_h\boldsymbol{u}^n-\boldsymbol{u}^n]+[\boldsymbol{u}^n-\boldsymbol{u}(t_n)]=\boldsymbol{W}_{\boldsymbol{u}}^n+\boldsymbol{I}_{\boldsymbol{u}}^n+\boldsymbol{e}_{\boldsymbol{u}}^n,\\ F_\eta^n&=[\eta_h^n-R_h\eta^n]+[R_h\eta^n-\eta^n]+[\eta^n-\eta(t_n)]=W_\eta^n+I_\eta^n+e_\eta^n,\end{aligned}$$

it suffices to $\boldsymbol{W}_{\boldsymbol{u}}^n$, $\boldsymbol{I}_{\boldsymbol{u}}^n$, W_η^n, and I_η^n which are defined self-explanatorily above.

Lemma 4.4. *Let the assumptions of Lemma 4.2 be satisfied. There exists a constant $C>0$ depending only on T, $\|\boldsymbol{f}\|_{\boldsymbol{C}^1(0,T;\boldsymbol{L}^2(\Omega))}$ and the material parameters such that, for all $(\boldsymbol{v}_h,\varphi_h)\in\boldsymbol{X}_h\times Y_h$,*

$$\|\boldsymbol{W}_{\boldsymbol{u}}^n\|_{l^\infty(0,T;\boldsymbol{L}^2(\Omega))}+\|\boldsymbol{W}_{\boldsymbol{u}}^n\|_{l^2(0,T;\boldsymbol{X})}\le C\|\boldsymbol{I}_{\boldsymbol{u}}^n\|_{l^\infty(0,T;\boldsymbol{L}^2(\Omega))}+C\|\boldsymbol{I}_{\boldsymbol{u}}^n\|_{l^2(0,T;\boldsymbol{X})}, \tag{4.29a}$$

$$\begin{aligned}\left\|W_\eta^n\right\|_{l^\infty(0,T;L^2(\Omega_c))}+\left\|W_\eta^n\right\|_{l^2(0,T;Y)}\le{}& C\left[\|\boldsymbol{I}_{\boldsymbol{u}}^n\|_{l^\infty(0,T;\boldsymbol{L}^2(\Omega))}+\|\boldsymbol{I}_{\boldsymbol{u}}^n\|_{l^2(0,T;\boldsymbol{X})}\right]\\ &+C\left[\left\|I_\eta^n\right\|_{l^\infty(0,T;L^2(\Omega_c))}+\left\|I_\eta^n\right\|_{l^2(0,T;Y)}\right].\end{aligned} \tag{4.29b}$$

Proof. Write $\boldsymbol{S}_{\boldsymbol{u}}^n=\boldsymbol{u}_h^n-\boldsymbol{u}^n$ and $S_\eta^n=\eta_h^n-\eta^n$. From (4.1) and (4.24), we easily get the Galerkin orthogonality

$$\begin{aligned}&\int_\Omega \sigma\left(\boldsymbol{S}_{\boldsymbol{u}}^n-\boldsymbol{S}_{\boldsymbol{u}}^{n-1}\right)\cdot\boldsymbol{v}_h+\tau\int_\Omega \boldsymbol{\nu}^{n-1}\mathbf{curl}\,\boldsymbol{S}_u^n\cdot\mathbf{curl}\,\boldsymbol{v}_h=\tau\int_\Omega(\boldsymbol{\nu}^{n-1}-\boldsymbol{\nu}_h^{n-1})\,\mathbf{curl}\,\boldsymbol{u}^n\,\mathbf{curl}\,\boldsymbol{v}_h,\\ &\int_{\Omega_c} C_\rho(S_\eta^n-S_\eta^{n-1})\varphi_h+\tau a(\lambda^{n-1},S_\eta^n,\varphi_h)=\tau\int_{\Omega_c}\sigma\boldsymbol{S}_{\boldsymbol{u}}^n\cdot(\boldsymbol{S}_{\boldsymbol{u}}^n+2\boldsymbol{u}^n)\varphi_h+\tau\int_{\Gamma_c}(\lambda^n-\lambda_h^n)\eta^n\varphi_h,\end{aligned}$$

for all $(\boldsymbol{v}_h,\varphi_h)\in\boldsymbol{X}_h\times Y_h$. It is easy to see that the first equality also holds for all $\boldsymbol{v}_h\in\boldsymbol{V}_h$. Therefore, we can take $\boldsymbol{v}_h=\boldsymbol{W}_{\boldsymbol{u}}^n$ and $\varphi_h=W_\eta^n$ in the two equalities. By Schwarz's inequality and arguments similar to the proof of Theorem 4.1, we get (4.29). The details are omitted here. □

Theorem 4.2. *Let the assumptions of Lemma* 4.2 *be satisfied and let* $(\boldsymbol{u},\eta)$, $(\boldsymbol{u}_h^n,\eta_h^n)$ *be the solutions of* (3.6) *and* (4.24) *respectively. For sufficiently small* τ, *there exists a constant* $C>0$ *depending on* $\|\boldsymbol{u}\|_{\boldsymbol{C}^2(0,T;\boldsymbol{H}^{k+1}(\Omega))}$, $\|\eta\|_{C^2(0,T;H^{k+1}(\Omega_c))}$, *but independent of* h, τ *such that*

$$\begin{aligned}&\|\boldsymbol{F}_{\boldsymbol{u}}^n\|_{l^\infty(0,T;\boldsymbol{L}^2(\Omega))}+h\|\mathbf{curl}\,\boldsymbol{F}_{\boldsymbol{u}}^n\|_{l^2(0,T;\boldsymbol{L}^2(\Omega))}\le C(\tau+h^k),\\ &\|F_\eta^n\|_{l^\infty(0,T;L^2(\Omega_c))}+h\|\nabla F_\eta^n\|_{l^2(0,T;L^2(\Omega_c))}\le C(\tau+h^{\min(k+1,l)}).\end{aligned}$$

Proof. The proof is a direct consequence of Theorem 4.1, (4.27), (4.28), and Lemma 4.4. □

5 Numerical experiment

In this section, we present validate the magneto-heat coupling model by computing an engineering benchmark problem, TEAM Problem 21^b-MN [7]. We refer to [8] for more details about this model. The parallel code is developed based on the finite element package parallel Hierarchical Grids (PHG) [24]. All computations are carried out on the LSSC-IV Cluster of the State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences.

The geometry for P21^b-MN is shown in Fig. 3. The model consists of a magnetic steel and a nonmagnetic steel. The applied electric currents are carried by two coils and flow in opposite directions. The radiation condition for the temperature is set by

$$-\partial_{\boldsymbol{n}_c}\eta=\eta+\eta^4\quad\text{on }\Gamma_c.$$

The magnetic resistivity $\boldsymbol{\nu}$ is determined by the B-H curves given in [8]. The electric conductivities for the magnetic and nonmagnetic steels are specified in [8].

Since we are investigating the error between the solutions of (4.24) and the solutions of (3.6), to reduce the numerical error sufficiently, we adopt a fine mesh of Ω with 6×10^6 tetrahedra and 1.5×10^8 degrees of freedom. Fig. 4 shows that the calculated values of the magnetic flux agree well with the experimental values [7]. Fig. 5 shows the current density and temperature distribution at the end time $T=10s$. They indicate that the original problem (1.1)-(1.2) provides an accurate model for the magnetic-heat coupling system and the finite element method is efficient to solve the nonlinear problem.

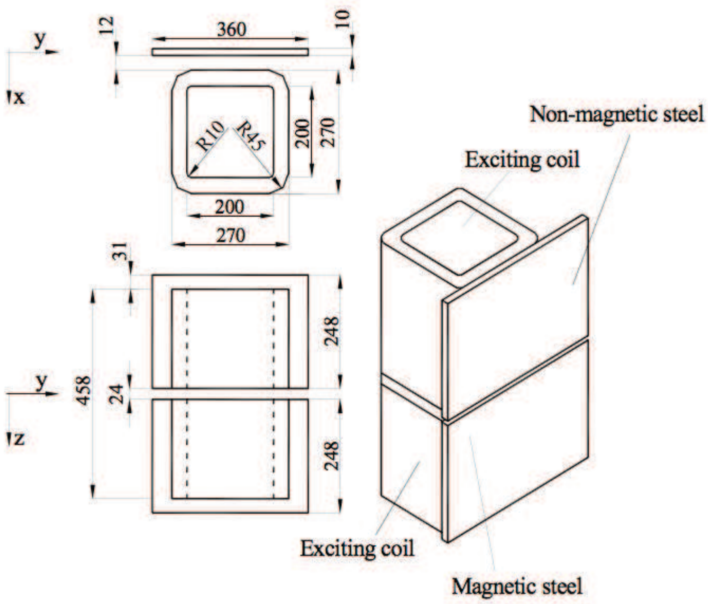

Figure 3: Geometry of Problem P21^b-MN.

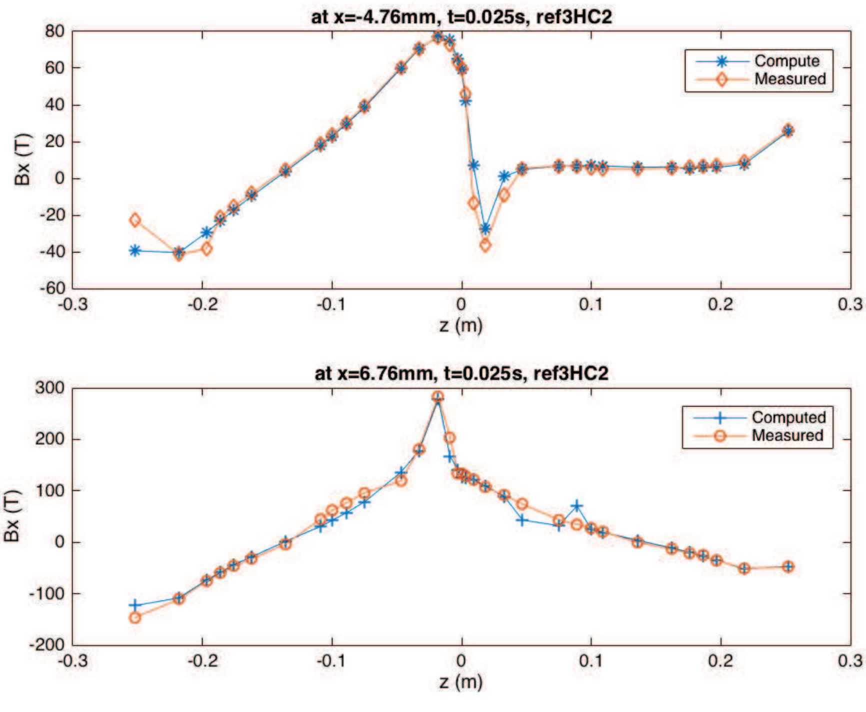

Figure 4: Numerical and experimental values of B_x at two lines $(4.76,0,z)$ and $(6.76,0,z)$.

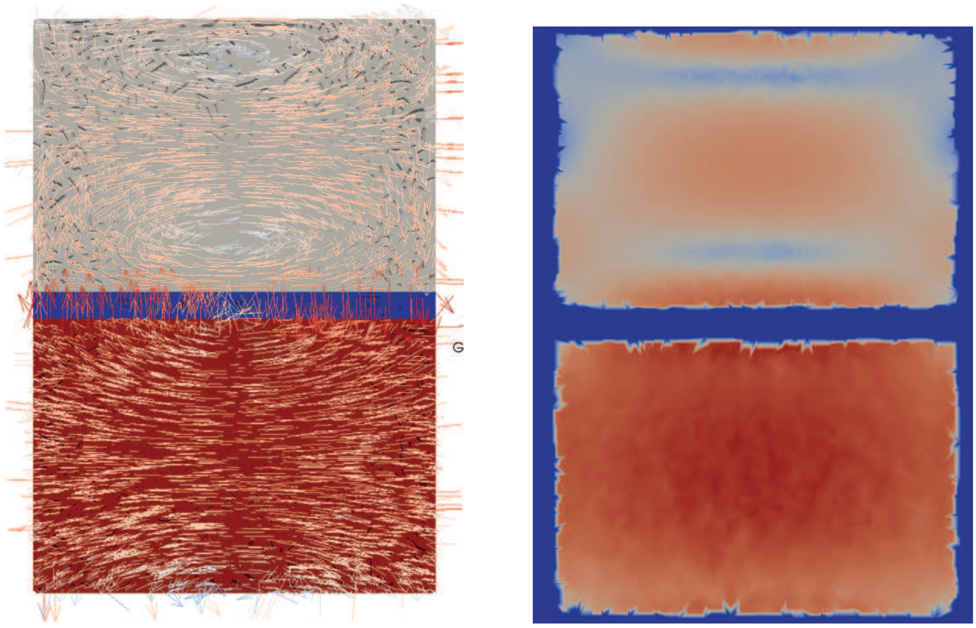

Figure 5: The current density and temperature distribution at plane $x=5$ mm.

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