Commun. Comput. Phys.
doi: 10.4208/cicp.2019.js60.12

Vol. **26**, No. 5, pp. 1510-1529
November 2019

Convergence Analysis of Exponential Time Differencing Schemes for the Cahn-Hilliard Equation[†]

Xiao Li[1], Lili Ju[1,*] and Xucheng Meng[1]

[1] *Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA.*

Received 15 March 2019; Accepted (in revised version) 1 May 2019

Abstract. In this paper, we rigorously prove the convergence of fully discrete first- and second-order exponential time differencing schemes for solving the Cahn-Hilliard equation. Our analyses mainly follow the standard procedure with the consistency and stability estimates for numerical error functions, while the technique of higher-order consistency analysis is adopted in order to obtain the uniform L^∞ boundedness of the numerical solutions under some moderate constraints on the time step and spatial mesh sizes. This paper provides a theoretical support for numerical analysis of exponential time differencing and other related numerical methods for phase field models, in which an assumption on the uniform L^∞ boundedness is usually needed.

AMS subject classifications: 35K55, 65M12, 65M15, 65F30

Key words: Cahn-Hilliard equation, exponential time differencing, convergence analysis, uniform L^∞ boundedness.

1 Introduction

In this paper, we consider the Cahn-Hilliard equation [5],

$$u_t = -\varepsilon^2 \Delta^2 u + \Delta f(u), \quad \boldsymbol{x} \in \Omega,\ t \in (0,T] \tag{1.1}$$

with $f(u) = u^3 - u$, where Ω is a rectangle in $\mathbb{R}^2$ or a cuboid in $\mathbb{R}^3$ and $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is the unknown function subject to the periodic boundary condition. An important feature of the Cahn-Hilliard equation (1.1) is that it can be regarded as the H^{-1} gradient flow with respect to the Ginzburg-Landau energy functional

$$E(u) = \int_\Omega \Big(\frac{\varepsilon^2}{2} |\nabla u|^2 + F(u) \Big) \mathrm{d}\boldsymbol{x} \tag{1.2}$$

[†]This paper is dedicated to Professor Jie Shen on the occasion of his 60th birthday.
[*]Corresponding author. *Email addresses:* `li434@mailbox.sc.edu` (X. Li), `ju@math.sc.edu` (L. Ju), `mengx@mailbox.sc.edu` (X. Meng)

with $F(u)=\frac{1}{4}(u^2-1)^2$, and thus the solution of (1.1) satisfies the energy law (noting that $f(u)=F'(u)$):

$$\frac{\mathrm{d}E(u)}{\mathrm{d}t}=-\int_\Omega |\nabla(-\varepsilon^2\Delta u+f(u))|^2\mathrm{d}\boldsymbol{x}\leq 0, \tag{1.3}$$

i.e., the energy E is decreasing along the time. As one of the typical systems for phase field modeling, the Cahn-Hilliard equation has been widely used to model the phase separations and accumulation occurring in mixtures of small molecules and some other moving interface problems involving mass-conserved order parameters (see, e.g., [2,3,6, 10,33]). Thus, accurate and stable temporal discretizations for the Cahn-Hilliard equation are important for large scale and long-time simulations of coarsening dynamics.

In recent years, numerous numerical methods have been proposed for solving the Cahn-Hilliard equation and many other phase field models, in which the discrete version of the energy law (1.3) attracts much attention in numerical analysis. The modified Crank-Nicolson scheme [11,17,41] and the convex splitting scheme [14] are proven to be unconditionally energy stable (see also [34,35,39]). These schemes are usually nonlinear so that they are time-consuming due to the need of nonlinear solvers in each time step.

To avoid the nonlinear iterations, a first-order linear scheme was constructed in [21] by adding an extra stabilization term to the classic semi-implicit scheme for the Cahn-Hilliard equation. This stabilized scheme is unconditionally energy stable if the stabilizing parameter satisfies some certain inequality which depends on the L^∞ bound of the numerical solutions. Such scheme has been also applied to some other phase field models [8,40] with the same assumptions needed. Later, the first- and second-order stabilized schemes were studied more systematically in [16,37] under an assumption on the Lipschitz continuity of the nonlinear term $f(u)$. Since $f(u)$ is a polynomial of degree three in (1.1), the assumption on the Lipschitz continuity of $f(u)$ is in fact equivalent to the uniform L^∞ boundedness of the numerical solutions. Recently, an exponential time differencing (ETD) method for the Cahn-Hilliard equation was proposed in [28] based on the same stabilizing technique, and the stabilizing parameter is also required to depend on the numerical solutions to guarantee the energy stability. In addition, the classic backward Euler scheme was analyzed in [13] and the error estimate was also derived by assuming the uniform L^∞ boundedness of the numerical solutions.

Since the energy law (1.3) in the PDE level holds without any extra requirements, it is highly desired to remove the L^∞ assumption on the numerical solutions for theoretical completeness of stability and convergence analysis. By using advanced harmonic analysis for the stabilized scheme, the authors of [32] removed the technical restrictions and established the unconditional energy stability of the stabilized scheme for general phase field models. The energy stability of the second-order stabilized scheme in 2D and 3D spaces was analyzed in [30, 31] by using the similar approach. On the other hand, some investigations were devoted to the theoretical justification of the uniform L^∞ boundedness. It was shown in [4] that for a truncated potential $F(u)$ with quadratic growth at infinities, the maximum norm of the solution of the Cahn-Hilliard equation is bounded. The technique with a truncated potential was also considered in the literature, e.g., [36, 38]. In [12], a high-order regularity of the solution of the Cahn-Hilliard equation was obtained from the energy law and the L^∞ bound was derived by the Sobolev embedding theorem. Similar approaches were applied to the numerical analysis for the Cahn-Hilliard equation [1, 15] and for other phase field models [26, 39]. Another technique to justify the uniform L^∞ bound resorts to the convergence analysis and the view, adopted first in [18], that the numerical solutions are indeed the exact solution plus small perturbations, see [19, 20], which also is the crucial tool we will adopt in this paper.

The main contribution of this paper is to provide a rigorous convergence analysis for the first- and second-order ETD schemes for the Cahn-Hilliard equation and the theoretical justification of the uniform L^∞ bound of the numerical solutions. The ETD schemes [7, 24, 25, 27] come from the variation-of-constants formula with the nonlinear terms in the system approximated by polynomial interpolations, followed by exact integration of the resulted integrals. Combined with linear stabilization techniques, the ETD schemes can be efficiently implemented via fast Fourier transform (FFT) for problems in rectangular domains, which leads to many successful applications to simulations of coarsening dynamics, see, e.g., [9, 26, 28, 29, 42]. The convergence analysis would follow the standard procedure of consistency and stability analysis of the numerical errors. In order to obtain the uniform L^∞ boundedness of the numerical solutions under some moderate constraints on the time step size, we will carry out the higher-order consistency analysis.

The rest of this paper is organized as follows. In Section 2, we first present the fully discrete ETD schemes and some preliminary lemmas to be used in the later sections. Convergence analyses of the first- and second-order ETD schemes are then carried out in Sections 3 and 4 respectively, along with the proofs of the uniform L^∞ boundedness of the numerical solutions. Finally, some concluding remarks are given in Section 5.

2 Fully discrete exponential time differencing schemes

For simplicity, in this paper, we restrict our analyses to the 2D case with $\Omega=(0,L_x)\times(0,L_y)$, and the results for 3D case could be obtained similarly without difficulties.

2.1 Spatial discretization

Given two positive integer N_x and N_y, define the index set

$$S_h=\{(i,j)\in\mathbb{Z}^2 \,|\, 1\le i\le N_x,\ 1\le j\le N_y\}.$$

The $N_x\times N_y$ mesh Ω_h of the domain Ω is a set of nodes (x_i,y_j) with $x_i=ih_x$, $y_j=jh_y$, $(i,j)\in S_h$, where $h_x=L_x/N_x$ and $h_y=L_y/N_y$ are the uniform mesh sizes in each direction. Let $h=\max\{h_x,h_y\}$ and $h_0=\min\{h_x,h_y\}$. We always require that there exists a fixed constant C_0 such that the mesh satisfies $h\le C_0h_0$. Corresponding to the periodic boundary condition, denote by $\mathcal{M}_h$ all the periodic grid functions defined on Ω_h, i.e.,

$$\mathcal{M}_h=\{v:\Omega_h\to\mathbb{R}\,|\,v_{i+mN_x,j+nN_y}=v_{ij}\text{ for any }(i,j)\in S_h\text{ and }(m,n)\in\mathbb{Z}^2\}.$$

For any $v,w\in\mathcal{M}_h$, the discrete L^2 inner product $\langle\cdot,\cdot\rangle$, discrete L^2 norm $\|\cdot\|$, and discrete L^∞ norm $\|\cdot\|_\infty$ are respectively defined by

$$\langle v,w\rangle=h_xh_y\sum_{(i,j)\in S_h}v_{ij}w_{ij},\qquad \|v\|=\sqrt{\langle v,v\rangle},\qquad \|v\|_\infty=\max_{(i,j)\in S_h}|v_{ij}|.$$

For any $v\in\mathcal{M}_h$, we call $\overline{v}:=\frac{1}{N_xN_y}\langle v,1\rangle$ the mean value of v. In particular, denote by $\mathcal{M}_h^0$ all the grid functions in $\mathcal{M}_h$ with mean zero, i.e., $\mathcal{M}_h^0=\{v\in\mathcal{M}_h\,|\,\langle v,1\rangle=0\}$. For any $v\in\mathcal{M}_h$, the discrete Laplace operator A is defined by

$$(Av)_{ij}=\frac{1}{h_x^2}(v_{i-1,j}-2v_{ij}+v_{i+1,j})+\frac{1}{h_y^2}(v_{i,j-1}-2v_{ij}+v_{i,j+1}).$$

Lemma 2.1. (i) *A is self-adjoint and negative semi-definite on $\mathcal{M}_h$, i.e.,*

$$\langle v,Aw\rangle=\langle Av,w\rangle,\quad \langle Av,v\rangle\le 0,\qquad \forall v,w\in\mathcal{M}_h.$$

(ii) *A is self-adjoint and negative definite, and thus, invertible on $\mathcal{M}_h^0$.*

The space-discrete scheme for (1.1) is to find a function $U:[0,\infty)\to\mathcal{M}_h^0$ such that

$$\begin{cases}\dfrac{\mathrm{d}U}{\mathrm{d}t}+\varepsilon^2A^2U=Af(U),\quad t\in(0,T],\\ U(0)=U_0,\end{cases}\tag{2.1}$$

where $U_0\in\mathcal{M}_h^0$ is given by the initial data and $(f(U))_{ij}=U_{ij}^3-U_{ij}$ for any $(i,j)\in S_h$. For the sake of the stability of the time-stepping schemes, we introduce a stabilizing parameter $\kappa>0$ and rewrite the equation in (2.1) as

$$\frac{\mathrm{d}U}{\mathrm{d}t}+L_hU=Af_\kappa(U),\quad t\in(0,T],\tag{2.2}$$

where $L_h:=\varepsilon^2A^2-\kappa A$ and $f_\kappa(U):=f(U)-\kappa U$. According to Lemma 2.1-(ii), the linear part L_h is self-adjoint and positive definite on $\mathcal{M}_h^0$.

2.2 Temporal integration

Given a positive integer $M_t>0$, we partition the time interval $[0,T]$ by $\{t_m=m\tau:0\le m\le M_t\}$ with a uniform time step size $\tau=T/M_t$. By using the exponential integrator, the exact solution of (2.2) satisfies the variation-of-constants formula:

$$U(t_{m+1})=\mathrm{e}^{-\tau L_h}U(t_m)+\int_0^\tau \mathrm{e}^{-(\tau-s)L_h}Af_\kappa(U(t_m+s))\,\mathrm{d}s,\tag{2.3}$$

for $m=0,1,\cdots,M_t-1$. The key idea to develop the ETD schemes from (2.3) is to approximate the nonlinear part $Af_\kappa(U(t_m+s))$ by lower-order polynomials in $s\in(0,\tau)$ and to integrate the resulted integral exactly. We denote by U^m the approximation of $U(t_m)$ generated by the ETD schemes and introduce several exponential-related functions:

$$\phi_{-1}(a)=\mathrm{e}^{-a},\quad \phi_0(a)=\frac{1-\mathrm{e}^{-a}}{a},\quad \phi_1(a)=\frac{a-1+\mathrm{e}^{-a}}{a^2},\qquad a\neq 0. \tag{2.4}$$

Case 1. Approximating $Af_\kappa(U(t_m+s))$ by the constant $Af_\kappa(U(t_m))$, we obtain the first-order ETD (ETD1) scheme:

$$U^{m+1}=\phi_{-1}(\tau L_h)U^m+\tau\phi_0(\tau L_h)Af_\kappa(U^m), \tag{2.5}$$

for $m=0,1,\cdots,M_t-1$.

Case 2. Approximating $Af_\kappa(U(t_m+s))$ by a linear extrapolation based on $Af_\kappa(U(t_m))$ and $Af_\kappa(U(t_{m-1}))$, we obtain the second-order ETD multistep (ETDMs2) scheme:

$$U^{m+1}=\phi_{-1}(\tau L_h)U^m+\tau[(\phi_0+\phi_1)(\tau L_h)Af_\kappa(U^m)-\phi_1(\tau L_h)Af_\kappa(U^{m-1})], \tag{2.6}$$

for $m=1,2,\cdots,M_t-1$, where U^1 is calculated by the ETD1 scheme.

Case 3. Approximating $Af_\kappa(U(t_m+s))$ by a linear interpolation based on $Af_\kappa(U(t_m))$ and $Af_\kappa(U(t_{m+1}))$, we obtain the second-order ETD Runge-Kutta (ETDRK2) scheme:

$$U^{m+1}=\phi_{-1}(\tau L_h)U^m+\tau[(\phi_0-\phi_1)(\tau L_h)Af_\kappa(U^m)+\phi_1(\tau L_h)Af_\kappa(\widetilde{U}^{m+1})], \tag{2.7}$$

for $m=0,1,\cdots,M_t-1$, where $\widetilde{U}^{m+1}$ is an approximation of $U(t_{m+1})$, given by the ETD1 solution.

Numerical simulations of the coarsening dynamics based on the Cahn-Hilliard equation (1.1) were carried out in [28] by using the ETD1 scheme (2.5) and the ETDMs2 scheme (2.6). Without more discussions on numerical performances of the ETD schemes, we just point out that the practical efficiency of the ETD schemes proposed above depends highly on the implementation of the actions of the matrix exponential-related functions $\phi_\gamma(\tau L_h)$, $\gamma=-1,0,1$, such as FFT-based method and the Krylov subspace method [23].

2.3 Equivalent forms of the ETD schemes

According to the constructions of the ETD schemes, we now present their equivalent forms, which are more convenient to numerical analysis. For given $U^m,U^{m-1}\in\mathcal{M}_h^0$, the solution of the ETD scheme is actually given by $U^{m+1}=W(\tau)$ with the function $W:[0,\tau]\to\mathcal{M}_h^0$ determined by the evolution equation:

Case 1. For the ETD1 scheme:

$$\begin{cases}\dfrac{\mathrm{d}W(s)}{\mathrm{d}s}+L_hW(s)=Af_\kappa(U^m), \quad s\in(0,\tau),\\ W(0)=U^m.\end{cases} \tag{2.8}$$

Case 2. For the ETDMs2 scheme:

$$\begin{cases} \dfrac{\mathrm{d}W(s)}{\mathrm{d}s}+L_hW(s)=\Big(1+\dfrac{s}{\tau}\Big)Af_\kappa(U^m)-\dfrac{s}{\tau}Af_\kappa(U^{m-1}), \quad s\in(0,\tau), \\ W(0)=U^m. \end{cases} \tag{2.9}$$

Case 3. For the ETDRK2 scheme:

$$\begin{cases} \dfrac{\mathrm{d}W(s)}{\mathrm{d}s}+L_hW(s)=\Big(1-\dfrac{s}{\tau}\Big)Af_\kappa(U^m)+\dfrac{s}{\tau}Af_\kappa(\widetilde{U}^{m+1}), \quad s\in(0,\tau), \\ W(0)=U^m, \end{cases} \tag{2.10}$$

where $\widetilde{U}^{m+1}$ is given by the ETD1 solution.

2.4 Preliminary lemmas

To carry out the convergence analysis, we define the norm $\|\!|Q|\!\|$ for a self-adjoint linear operator $Q:\mathcal{M}_h\to\mathcal{M}_h$ by the spectrum radius of Q, then it obviously holds

$$\|Qv\|\le\|\!|Q|\!\|\|v\|, \quad \forall v\in\mathcal{M}_h.$$

Since $\mathcal{M}_h$ is a finite-dimensional linear space, the following properties of matrix functions could be utilized on the linear operators A and L_h.

Lemma 2.2 (see [22]). *Let ϕ be defined on the spectrum of $M\in\mathbb{C}^{n\times n}$, that is, the values*

$$\phi^{(j)}(\lambda_i), \quad 0\le j\le n_i-1, \quad 1\le i\le n$$

exist, where $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of M and n_i is the order of the largest Jordan block where λ_i appears. Then

(i) *$\phi(M)$ commutes with M;*

(ii) *$\phi(M^T)=\phi(M)^T$;*

(iii) *the eigenvalues of $\phi(M)$ are $\{\phi(\lambda_i)\}_{i=1}^n$.*

Some simple and useful properties of the exponential-related functions defined in (2.4) are collected in the following lemma without proof.

Lemma 2.3. (i) *For any $a>0$, the following inequalities hold:*

$$\begin{aligned} &0<(1+a)\phi_{-1}(a)<1, \\ &1<(1+a)\phi_0(a)<2, \\ &\frac{1}{2}<(1+a)\phi_1(a)<1, \\ &0<(1+a)[\phi_0(a)-\phi_1(a)]<1. \end{aligned}$$

(ii) *If $0<s<\tau\le 1$, then, for any $a>0$, it holds*

$$0<(1+a\tau)\mathrm{e}^{-a(\tau-s)}<1.$$

To obtain the L^∞ boundedness of the numerical solutions, the following discrete inverse inequalities play an important role. The proof is a direct application of the definitions of the discrete norms and thus we omit it.

Lemma 2.4 (Discrete inverse inequality). *For any $v\in\mathcal{M}_h$, it always holds that*

$$\|v\|_\infty \le C_{inv} h^{-1}\|v\| \qquad \text{(in 2D case)}, \tag{2.11}$$

or

$$\|v\|_\infty \le C_{inv} h^{-\frac{3}{2}}\|v\| \qquad \text{(in 3D case)}, \tag{2.12}$$

where $C_{inv}>0$ is a constant independent of h.

3 Convergence analysis of the first-order scheme

In this and next sections, we will present the convergence analysis for the ETD schemes along with the uniform L^∞ boundedness of the numerical solutions. We define the operator I^h to limit a function $v:\overline{\Omega}\to\mathbb{R}$ on the mesh Ω_h such that

$$I^h v\in\mathcal{M}_h \text{ with } (I^h v)_{ij} = v(x_i,y_j), \quad \forall v\in C(\overline{\Omega}).$$

Denote by $u_e(x,y,t)$ the exact solution of (1.1) and define $U_e(t)=I^h u_e(t)$ for each $t\in[0,T]$, i.e., $(U_e)_{ij}(t)=u_e(x_i,y_j,t)$, to represent its corresponding grid function in the spatial mesh. Let

$$B := \|u_e\|_{L^\infty(\Omega\times(0,T))}+1<\infty,$$

due to the smoothness of u_e. The numerical solutions of the ETD schemes (2.5), (2.6) and (2.7) will be proved to be L^∞ bounded uniformly in the sense that

$$\|U^m\|_\infty \le B, \quad 0\le m\le M_t, \tag{3.1}$$

for small sufficiently spatial mesh and time step sizes.

Let us begin with the first-order scheme. The following theorem is the main result of this section and the detailed proof is given in the following three subsections.

Theorem 3.1. *Suppose the solution u_e of the Cahn-Hilliard equation* (1.1) *is smooth enough and $\{U^m\}_{0\le m\le M_t}$ is the solution of the ETD1 scheme* (2.5). *If τ and h are small sufficiently and under the linear refinement path constraint $\tau\le C'h$ with $C'>0$ any fixed constant, we have the uniform L^∞ bound* (3.1) *and the error estimate*

$$\|U^m - U_e(t_m)\| \le C(\tau+h^2), \quad 0\le m\le M_t,$$

where the constant $C>0$ is independent of τ and h.

We prove it by induction. Suppose the numerical solutions U^k, $k=0,1,\cdots,m$ are known and we make an L^∞ assumption for these numerical solutions, i.e.,

$$\|U^k\|_\infty \le B, \quad 0\le k\le m. \tag{3.2}$$

We will estimate the error between U^{m+1} and $U_e(t_{m+1})$, and then, recover the L^∞ bound for U^{m+1} as (3.2) when the time step and spatial mesh sizes are small sufficiently.

3.1 Error estimate and L^∞ boundedness

Define a function $W_e(s)=U_e(t_m+s)$ for $s\in[0,\tau]$, then, by consistency, W_e satisfies

$$\begin{cases}\dfrac{\mathrm{d}W_e(s)}{\mathrm{d}s}+L_hW_e(s)=Af_\kappa(U_e(t_m))+R_m^{(1)}(s), \quad s\in(0,\tau),\\ W_e(0)=U_e(t_m),\end{cases} \tag{3.3}$$

where $R_m^{(1)}(s)$ is the truncated error:

$$\begin{aligned}R_m^{(1)}(s)=&\,\varepsilon^2A^2U_e(t_m+s)-\varepsilon^2I^h(\Delta^2u_e)(t_m+s)-\kappa AU_e(t_m+s)+\kappa I^h(\Delta u_e)(t_m+s)\\&+I^h\Delta f_\kappa(u_e(t_m+s))-Af_\kappa(U_e(t_m+s))+Af_\kappa(U_e(t_m+s))-Af_\kappa(U_e(t_m)),\end{aligned}$$

and satisfies

$$\max_{0\le m\le M_t-1}\sup_{s\in(0,\tau)}\|R_m^{(1)}(s)\|\le C_1(\tau+h^2)$$

with C_1 depending on u_e, ε, κ, T and Ω. Let $e(s):=W(s)-W_e(s)$ with $W(s)$ determined by (2.8). The difference between (3.3) and (2.8) yields

$$\begin{cases}\dfrac{\mathrm{d}e(s)}{\mathrm{d}s}+L_he(s)=Af_\kappa(U^m)-Af_\kappa(U_e(t_m))-R_m^{(1)}(s), \quad s\in(0,\tau),\\ e(0)=U^m-U_e(t_m)=:e^m\in\mathcal{M}_h^0,\end{cases}$$

which gives the solution $e^{m+1}:=e(\tau)$ as

$$e^{m+1}=\phi_{-1}(\tau L_h)e^m+\tau\phi_0(\tau L_h)[Af_\kappa(U^m)-Af_\kappa(U_e(t_m))]-\int_0^\tau \mathrm{e}^{-(\tau-s)L_h}R_m^{(1)}(s)\,\mathrm{d}s. \tag{3.4}$$

Acting $I+\tau L_h$ on both sides of (3.4) and taking the discrete L^2 inner product of the resulted equation with e^{m+1} yield

$$\|e^{m+1}\|^2+\varepsilon^2\tau\|Ae^{m+1}\|^2-\kappa\tau\langle Ae^{m+1},e^{m+1}\rangle=\mathrm{RHS}, \tag{3.5}$$

where

$$\begin{aligned}\mathrm{RHS}=&\,\langle(I+\tau L_h)\phi_{-1}(\tau L_h)e^m,e^{m+1}\rangle\\&+\tau\langle(I+\tau L_h)\phi_0(\tau L_h)[f_\kappa(U^m)-f_\kappa(U_e(t_m))],Ae^{m+1}\rangle\\&-\int_0^\tau\langle(I+\tau L_h)\mathrm{e}^{-(\tau-s)L_h}R_m^{(1)}(s),e^{m+1}\rangle\,\mathrm{d}s.\end{aligned}$$

According to the assumption (3.2) and noting that $\|U_e(t_m)\|_\infty\le B$, we derive

$$\|f_\kappa(U^m)-f_\kappa(U_e(t_m))\|\le\|f_\kappa'(\xi_m)\|_\infty\|e^m\|\le|3B^2-\kappa-1|\|e^m\|$$

with ξ_m between U^m and $U_e(t_m)$ componentwisely. Using Lemma 2.3, we derive

$$\begin{aligned}\mathrm{RHS}\le&\,|||(I+\tau L_h)\phi_{-1}(\tau L_h)|||\,\|e^m\|\,\|e^{m+1}\|\\&+\tau|||(I+\tau L_h)\phi_0(\tau L_h)|||\,\|f_\kappa(U^m)-f_\kappa(U_e(t_m))\|\,\|Ae^{m+1}\|\\&+\int_0^\tau|||(I+\tau L_h)\mathrm{e}^{-(\tau-s)L_h}|||\mathrm{d}s\sup_{t\in(0,\tau)}\|R_m^{(1)}(t)\|\,\|e^{m+1}\|\\\le&\,\|e^m\|\,\|e^{m+1}\|+2\tau|3B^2-\kappa-1|\,\|e^m\|\,\|Ae^{m+1}\|+\tau\sup_{t\in(0,\tau)}\|R_m^{(1)}(t)\|\,\|e^{m+1}\|\\\le&\,\frac{1}{2}\|e^m\|^2+\frac{1}{2}\|e^{m+1}\|^2+\frac{(3B^2-\kappa-1)^2}{\varepsilon^2}\tau\|e^m\|^2+\varepsilon^2\tau\|Ae^{m+1}\|^2\\&+\frac{1}{2}\tau\sup_{t\in(0,\tau)}\|R_m^{(1)}(t)\|^2+\frac{1}{2}\tau\|e^{m+1}\|^2.\end{aligned}\tag{3.6}$$

Combining with (3.5) and denoting $K:=\frac{(3B^2-\kappa-1)^2}{\varepsilon^2}$, we obtain

$$\frac{1}{2}\|e^{m+1}\|^2-\frac{1}{2}\|e^m\|^2\le K\tau\|e^m\|^2+\frac{1}{2}\tau\|e^{m+1}\|^2+\frac{1}{2}\tau\sup_{t\in(0,\tau)}\|R_m^{(1)}(t)\|^2.$$

Summing the above inequality from 0 to m and noting $e^0=0$ lead to

$$\frac{1}{2}\|e^{m+1}\|^2\le\Big(K+\frac{1}{2}\Big)\tau\sum_{k=1}^m\|e^k\|^2+\frac{1}{2}\tau\|e^{m+1}\|^2+\frac{1}{2}\tau\sum_{k=0}^m\sup_{t\in(0,\tau)}\|R_k^{(1)}(t)\|^2,$$

or equivalently,

$$(1-\tau)\|e^{m+1}\|^2\le(2K+1)\tau\sum_{k=1}^m\|e^k\|^2+\tau\sum_{k=0}^m\sup_{t\in(0,\tau)}\|R_k^{(1)}(t)\|^2.$$

When $\tau\le\frac{1}{2}$, we have

$$\begin{aligned}\|e^{m+1}\|^2&\le(4K+2)\tau\sum_{k=1}^m\|e^k\|^2+2\tau\sum_{k=0}^m\sup_{t\in(0,\tau)}\|R_k^{(1)}(t)\|^2\\&\le(4K+2)\tau\sum_{k=1}^m\|e^k\|^2+2TC_1^2(\tau+h^2)^2.\end{aligned}$$

Using the discrete Gronwall's inequality, we obtain

$$\|e^{m+1}\|^2\le 2TC_1^2\mathrm{e}^{(4K+2)T}(\tau+h^2)^2,$$

that is,

$$\|e^{m+1}\|\le\sqrt{2T}C_1\mathrm{e}^{(2K+1)T}(\tau+h^2).$$

To obtain the L^∞ boundedness of U^{m+1} as (3.2), by using the discrete inverse inequality (2.11), we have

$$\|e^{m+1}\|_\infty \le \frac{C_{\rm inv}}{h}\|e^{m+1}\| \le \sqrt{2T}C_{\rm inv}C_1 {\rm e}^{(2K+1)T}\Big(\frac{\tau}{h}+h\Big).$$

If $\tau \le C'h^2$ and $h \le [\sqrt{2T}C_{\rm inv}C_1{\rm e}^{(2K+1)T}(C'+1)]^{-1}$, we obtain $\|e^{m+1}\|_\infty \le 1$. Then, the L^∞ bound of the numerical solution U^{m+1} becomes

$$\|U^{m+1}\|_\infty \le \|U^{m+1}-U_e(t_{m+1})\|_\infty + \|U_e(t_{m+1})\|_\infty \le 1+\|U_e(t_{m+1})\|_\infty \le B.$$

Remark 3.1. For 3D case, using the discrete inverse inequality (2.12), one could obtain the same result as long as $\tau \le C'h^2$ and $h \le [\sqrt{2T}C_{\rm inv}C_1{\rm e}^{(2K+1)T}(C'+1)]^{-2}$.

Note that we enforce a quadratic refinement path constraint $\tau \le C'h^2$ for the recovery of the L^∞ bound of U^{m+1}. That is because we used the inverse inequality for e^{m+1} and the order of $\|e^{m+1}\|$ was only $\mathcal{O}(\tau+h^2)$. If we obtain a higher order for the truncated error, e.g., $\mathcal{O}(\tau^2+h^2)$, the quadratic constraint could be reduced to a linear one. To this end, we need to conduct the higher-order consistency analysis.

3.2 Higher-order consistency analysis

By consistency, the exact solution u_e satisfies the space-discrete equation

$$\frac{{\rm d}U_e}{{\rm d}t}+L_hU_e = Af_\kappa(U_e(t_m))+\tau I^h g^{(1)}+\mathcal{O}(\tau^2+h^2), \quad t\in(t_m,t_{m+1}], \tag{3.7}$$

where the function $g^{(1)}(x,y,t)$ is smooth enough and depends only on the partial derivatives of u_e. Define the temporal correction function U_τ by solving the following ODE system:

$$\frac{{\rm d}U_\tau}{{\rm d}t}+L_hU_\tau = A[f'_\kappa(U_e(t_m))U_\tau(t_m)]-I^h g^{(1)}, \quad t\in(t_m,t_{m+1}], \tag{3.8}$$

subject to the zero initial value. Existence and uniqueness of the solution of this linear system is standard and the solution U_τ depends only on u_e. Define a grid function $\widehat{U}=U_e+\tau U_\tau$. Multiplying (3.8) by τ and adding the result and (3.7) give

$$\frac{{\rm d}\widehat{U}}{{\rm d}t}+L_h\widehat{U} = Af_\kappa(\widehat{U}(t_m))+\mathcal{O}(\tau^2+h^2), \quad t\in(t_m,t_{m+1}], \tag{3.9}$$

where we have used the fact

$$f_\kappa(\widehat{U}(t_m)) = f_\kappa(U_e(t_m))+\tau f'_\kappa(U_e(t_m))U_\tau(t_m)+\mathcal{O}(\tau^2).$$

Since $\|U_\tau(t)\|_\infty \le C$ with C depending only on u_e, we have

$$\|\widehat{U}(t)-U_e(t)\|_\infty \le C\tau. \tag{3.10}$$

When τ is small sufficiently, we further have

$$\|\widehat{U}(t)\|_\infty \le \|U_e(t)\|_\infty+\frac{1}{2}. \tag{3.11}$$

3.3 L^∞ boundedness under a linear refinement path constraint

Define a function $\widehat{W}(s)=\widehat{U}(t_m+s)$ for $s\in[0,\tau]$, then $\widehat{W}$ satisfies

$$\begin{cases}\dfrac{\mathrm{d}\widehat{W}(s)}{\mathrm{d}s}+L_h\widehat{W}(s)=Af_\kappa(\widehat{U}(t_m))+\widehat{R}_m^{(1)}(s), & s\in(0,\tau),\\ \widehat{W}(0)=\widehat{U}(t_m),\end{cases}$$

where $\widehat{R}_m^{(1)}(s)$ is the truncated error, according to (3.9), satisfying

$$\max_{0\le m\le M_t-1}\sup_{s\in(0,\tau)}\|\widehat{R}_m^{(1)}(s)\|\le\widehat{C}_1(\tau^2+h^2)$$

with $\widehat{C}_1$ depending on u_e, ε, κ, T and Ω. Repeating the error analysis above, we reach

$$\|U^{m+1}-\widehat{U}(t_{m+1})\|\le\sqrt{2T}\widehat{C}_1\mathrm{e}^{(2K+1)T}(\tau^2+h^2).$$

Therefore, using (3.10), we obtain

$$\|U^{m+1}-U_e(t_{m+1})\|\le\|U^{m+1}-\widehat{U}(t_{m+1})\|+\|\widehat{U}(t_{m+1})-U_e(t_{m+1})\|\le C(\tau+h^2),$$

where C depends on u_e, ε, κ, T and Ω. Using the discrete inverse inequality (2.11), we have

$$\|U^{m+1}-\widehat{U}(t_{m+1})\|_\infty\le\frac{C_{\mathrm{inv}}}{h}\|U^{m+1}-\widehat{U}(t_{m+1})\|\le\sqrt{2T}C_{\mathrm{inv}}\widehat{C}_1\mathrm{e}^{(2K+1)T}\Big(\frac{\tau^2}{h}+h\Big).$$

If $\tau\le C'h$ and $h\le[2\sqrt{2T}C_{\mathrm{inv}}\widehat{C}_1\mathrm{e}^{(2K+1)T}(C'^2+1)]^{-1}$, we obtain

$$\|U^{m+1}-\widehat{U}(t_{m+1})\|_\infty\le\frac{1}{2}.$$

Then, using (3.11), the L^∞ bound of the numerical solution U^{m+1} becomes

$$\|U^{m+1}\|_\infty\le\|U^{m+1}-\widehat{U}(t_{m+1})\|_\infty+\|\widehat{U}(t_{m+1})\|_\infty\le\frac{1}{2}+\|U_e(t_m)\|_\infty+\frac{1}{2}\le B.$$

Remark 3.2. For 3D case, using the discrete inverse inequality (2.12), one could obtain the same result as long as $\tau\le C'h$ and $h\le[2\sqrt{2T}C_{\mathrm{inv}}\widehat{C}_1\mathrm{e}^{(2K+1)T}(C'^2+1)]^{-2}$.

Up to now, we have finished the complete proof of the error estimate and L^∞ boundedness of the ETD1 scheme (2.5). As an important application of the uniform L^∞ boundedness, one can show that the property (1.3), namely, the solution of (1.1) decreases the energy (1.2) monotonically with time, can be inherited by the ETD1 scheme (2.5) with respect to the discrete energy

$$E_h(v):=-\frac{\varepsilon^2}{2}\langle Av,v\rangle+\langle F(v),1\rangle,\quad v\in\mathcal{M}_h.$$

This result is given by the following corollary without proof since the proof is standard (see our previous works [9,26]).

Corollary 3.1. *Under the assumptions of Theorem* 3.1*, if the stabilizing parameter satisfies*

$$\kappa \geq \frac{3}{2}B^2 - \frac{1}{2},$$

the energy inequality $E_h(U^{m+1}) \leq E_h(U^m)$ is valid for $m=0,1,\cdots,M_t-1$.

4 Convergence analysis of the second-order schemes

Now we turn to consider the convergence and uniform L^∞ boundedness of the second-order ETD schemes (2.6) and (2.7). Note that the ETD1 formula is involved in both the multistep scheme (2.6) and the Runge-Kutta scheme (2.7). Thus, we first present some preliminary derivations based on the error analysis for the ETD1 scheme: (1) Letting $m=0$ in (3.4) gives

$$e^1 = -\int_0^\tau \mathrm{e}^{-(\tau-s)L_h} R_1^{(1)}(s)\,\mathrm{d}s,$$

and thus,

$$\|e^1\| \leq \int_0^\tau |\!|\!|\mathrm{e}^{-(\tau-s)L_h}|\!|\!|\,\mathrm{d}s \sup_{t\in(0,\tau)} \|R_1^{(1)}(t)\| \leq \tau \sup_{t\in(0,\tau)} \|R_1^{(1)}(t)\|. \tag{4.1}$$

This estimate will be used later in the convergence analysis for the ETDMs2 scheme. (2) In the last step in (3.6), if we use the estimate

$$\tau \sup_{t\in(0,\tau)} \|R_m^{(1)}(t)\|\|e^{m+1}\| \leq \tau^2 \sup_{t\in(0,\tau)} \|R_m^{(1)}(t)\|^2 + \frac{1}{4}\|e^{m+1}\|^2,$$

we can obtain from (3.5) that

$$\|e^{m+1}\|^2 \leq (2+4K\tau)\|e^m\|^2 + 4\tau^2 \sup_{t\in(0,\tau)} \|R_m^{(1)}(t)\|^2. \tag{4.2}$$

This estimate will be used later in the convergence analysis for the ETDRK2 scheme.

The following theorem gives the results of both multistep and Runge-Kutta schemes and the proofs are carried out in the following two subsections, respectively.

Theorem 4.1. *Suppose the solution u_e of the Cahn-Hilliard equation* (1.1) *is smooth enough and $\{U^m\}_{0\leq m\leq M_t}$ is the solution of either the ETDMs2 scheme* (2.6) *or the ETDRK2 scheme* (2.7)*. If τ and h are small sufficiently and under the linear refinement path constraint $\tau \leq C'h$ with $C'>0$ any fixed constant, we have the uniform L^∞ bound* (3.1) *and the error estimate*

$$\|U^m - U_e(t_m)\| \leq C(\tau^2+h^2), \quad 0\leq m\leq M_t,$$

where the constant $C>0$ is independent of τ and h.

Again, we prove this theorem by induction and make the assumption (3.2), the L^∞ boundedness of the numerical solutions for the previous m steps.

4.1 Proof of Theorem 4.1 for the ETDMs2 scheme

The function $W_e(s)=U_e(t_m+s)$, $s\in[0,\tau]$ satisfies

$$\begin{cases}\dfrac{\mathrm{d}W_e(s)}{\mathrm{d}s}+L_hW_e(s)=\Big(1+\dfrac{s}{\tau}\Big)Af_\kappa(U_e(t_m))-\dfrac{s}{\tau}Af_\kappa(U_e(t_{m-1}))+R_m^{(2)}(s), \quad s\in(0,\tau),\\ W_e(0)=U_e(t_m),\end{cases} \tag{4.3}$$

where the truncated error $R_m^{(2)}(s)$ satisfies

$$\max_{0\le m\le M_t-1}\sup_{s\in(0,\tau)}\|R_m^{(2)}(s)\|\le C_2(\tau^2+h^2)$$

with C_2 depending on u_e, ε, κ, T and Ω. Let $e(s):=W(s)-W_e(s)$ with $W(s)$ determined by (2.9). The difference between (4.3) and (2.9) yields

$$\begin{cases}\dfrac{\mathrm{d}e(s)}{\mathrm{d}s}+L_he(s)=\Big(1+\dfrac{s}{\tau}\Big)[Af_\kappa(U^m)-Af_\kappa(U_e(t_m))]\\ \qquad\qquad\qquad -\dfrac{s}{\tau}[Af_\kappa(U^{m-1})-Af_\kappa(U_e(t_{m-1}))]-R_m^{(2)}(s), \quad s\in(0,\tau),\\ e(0)=U^m-U_e(t_m)=:e^m\in\mathcal{M}_h^0,\end{cases}$$

which gives the solution $e^{m+1}:=e(\tau)$ as

$$\begin{aligned}e^{m+1}=&\phi_{-1}(\tau L_h)e^m+\tau(\phi_0+\phi_1)(\tau L_h)[Af_\kappa(U^m)-Af_\kappa(U_e(t_m))]\\ &-\tau\phi_1(\tau L_h)[Af_\kappa(U^{m-1})-Af_\kappa(U_e(t_{m-1}))]-\int_0^\tau \mathrm{e}^{-(\tau-s)L_h}R_m^{(2)}(s)\,\mathrm{d}s.\end{aligned} \tag{4.4}$$

Acting $I+\tau L_h$ on both sides of (4.4) and taking the discrete L^2 inner product of the resulted equation with e^{m+1} yield

$$\|e^{m+1}\|^2+\varepsilon^2\tau\|Ae^{m+1}\|^2-\kappa\tau\langle Ae^{m+1},e^{m+1}\rangle=\mathrm{RHS}, \tag{4.5}$$

where

$$\begin{aligned}\mathrm{RHS}=&\langle(I+\tau L_h)\phi_{-1}(\tau L_h)e^m,e^{m+1}\rangle\\ &+\tau\langle(I+\tau L_h)(\phi_0+\phi_1)(\tau L_h)[f_\kappa(U^m)-f_\kappa(U_e(t_m))],Ae^{m+1}\rangle\\ &-\tau\langle(I+\tau L_h)\phi_1(\tau L_h)[f_\kappa(U^{m-1})-f_\kappa(U_e(t_{m-1}))],Ae^{m+1}\rangle\\ &-\int_0^\tau\langle(I+\tau L_h)\mathrm{e}^{-(\tau-s)L_h}R_m^{(2)}(s),e^{m+1}\rangle\,\mathrm{d}s.\end{aligned}$$

According to the assumption (3.2) and noting that $\|U_e(t)\|_\infty\le B$, we derive

$$\begin{aligned}&\|f_\kappa(U^m)-f_\kappa(U_e(t_m))\|\le|3B^2-\kappa-1|\|e^m\|,\\ &\|f_\kappa(U^{m-1})-f_\kappa(U_e(t_{m-1}))\|\le|3B^2-\kappa-1|\|e^{m-1}\|.\end{aligned}$$

Using Lemma 2.3, we derive

$$\begin{aligned}
\text{RHS} \le & \|(I+\tau L_h)\phi_{-1}(\tau L_h)\| \|e^m\| \|e^{m+1}\| \\
& +\tau \|(I+\tau L_h)(\phi_0+\phi_1)(\tau L_h)\| \|f_\kappa(U^m)-f_\kappa(U_e(t_m))\| \|Ae^{m+1}\| \\
& +\tau \|(I+\tau L_h)\phi_1(\tau L_h)\| \|f_\kappa(U^{m-1})-f_\kappa(U_e(t_{m-1}))\| \|Ae^{m+1}\| \\
& +\int_0^\tau \|(I+\tau L_h)\mathrm{e}^{-(\tau-s)L_h}\| \mathrm{d}s \sup_{t\in(0,\tau)} \|R_m^{(2)}(t)\| \|e^{m+1}\| \\
\le & \|e^m\| \|e^{m+1}\| + 3\tau|3B^2-\kappa-1| \|e^m\| \|Ae^{m+1}\| \\
& +\tau|3B^2-\kappa-1| \|e^{m-1}\| \|Ae^{m+1}\| + \tau \sup_{t\in(0,\tau)} \|R_m^{(2)}(t)\| \|e^{m+1}\| \\
\le & \frac{1}{2}\|e^m\|^2 + \frac{1}{2}\|e^{m+1}\|^2 + \frac{9(3B^2-\kappa-1)^2}{2\varepsilon^2}\tau\|e^m\|^2 + \frac{\varepsilon^2}{2}\tau\|Ae^{m+1}\|^2 \\
& + \frac{(3B^2-\kappa-1)^2}{2\varepsilon^2}\tau\|e^{m-1}\|^2 + \frac{\varepsilon^2}{2}\tau\|Ae^{m+1}\|^2 + \frac{1}{2}\tau \sup_{t\in(0,\tau)} \|R_m^{(2)}(t)\|^2 + \frac{1}{2}\tau\|e^{m+1}\|^2.
\end{aligned}$$

Combining with (4.5) and noting that $\frac{(3B^2-\kappa-1)^2}{\varepsilon^2}=K$, we obtain

$$\frac{1}{2}\|e^{m+1}\|^2 - \frac{1}{2}\|e^m\|^2 \le \frac{9K}{2}\tau\|e^m\|^2 + \frac{K}{2}\tau\|e^{m-1}\|^2 + \frac{1}{2}\tau\|e^{m+1}\|^2 + \frac{1}{2}\tau \sup_{t\in(0,\tau)} \|R_m^{(2)}(t)\|^2.$$

Summing the above inequality from 1 to m and noting $e^0=0$ lead to

$$\begin{aligned}
& \frac{1}{2}\|e^{m+1}\|^2 - \frac{1}{2}\|e^1\|^2 \\
\le & \frac{9K}{2}\tau\sum_{k=1}^m \|e^k\|^2 + \frac{K}{2}\tau\sum_{k=1}^m \|e^{k-1}\|^2 + \frac{1}{2}\tau\sum_{k=1}^m \|e^{k+1}\|^2 + \frac{1}{2}\tau\sum_{k=1}^m \sup_{t\in(0,\tau)} \|R_k^{(2)}(t)\|^2 \\
\le & \Big(5K+\frac{1}{2}\Big)\tau\sum_{k=1}^m \|e^k\|^2 + \frac{1}{2}\tau\|e^{m+1}\|^2 + \frac{1}{2}\tau\sum_{k=1}^m \sup_{t\in(0,\tau)} \|R_k^{(2)}(t)\|^2,
\end{aligned}$$

or equivalently,

$$(1-\tau)\|e^{m+1}\|^2 \le \|e^1\|^2 + (10K+1)\tau\sum_{k=1}^m \|e^k\|^2 + \tau\sum_{k=1}^m \sup_{t\in(0,\tau)} \|R_k^{(2)}(t)\|^2.$$

Here we use (4.1) to estimate $\|e^1\|^2$. When $\tau\le\frac{1}{2}$, we have

$$\begin{aligned}
\|e^{m+1}\|^2 \le & 2\|e^1\|^2 + (20K+2)\tau\sum_{k=1}^m \|e^k\|^2 + 2\tau\sum_{k=1}^m \sup_{t\in(0,\tau)} \|R_k^{(2)}(t)\|^2 \\
\le & (20K+2)\tau\sum_{k=1}^m \|e^k\|^2 + 2\tau^2 \sup_{t\in(0,\tau)} \|R_1^{(1)}(t)\|^2 + 2\tau\sum_{k=1}^m \sup_{t\in(0,\tau)} \|R_k^{(2)}(t)\|^2 \\
\le & (20K+2)\tau\sum_{k=1}^m \|e^k\|^2 + (2C_1^2+2TC_2^2)(\tau^2+h^2)^2.
\end{aligned}$$

Using the discrete Gronwall's inequality, we obtain

$$\|e^{m+1}\|^2 \le (2C_1^2+2TC_2^2)\mathrm{e}^{(20K+2)T}(\tau^2+h^2)^2,$$

that is,

$$\|e^{m+1}\| \le \sqrt{2C_1^2+2TC_2^2}\,\mathrm{e}^{(10K+1)T}(\tau^2+h^2).$$

Then, we use the discrete inverse inequality (2.11) to obtain

$$\|e^{m+1}\|_\infty \le \frac{C_{\mathrm{inv}}}{h}\|e^{m+1}\| \le \sqrt{2C_1^2+2TC_2^2}\,C_{\mathrm{inv}}\mathrm{e}^{(10K+1)T}\Big(\frac{\tau^2}{h}+h\Big).$$

If $\tau \le C'h$ and $h \le \big[\sqrt{2C_1^2+2TC_2^2}\,C_{\mathrm{inv}}\mathrm{e}^{(10K+1)T}(C'^2+1)\big]^{-1}$, we obtain $\|e^{m+1}\|_\infty \le 1$. Then, the L^∞ bound of the numerical solution U^{m+1} becomes

$$\|U^{m+1}\|_\infty \le \|U^{m+1}-U_e(t_{m+1})\|_\infty + \|U_e(t_{m+1})\|_\infty \le 1+\|U_e(t_{m+1})\|_\infty \le B.$$

This completes the proof of Theorem 4.1 for the multistep case.

4.2 Proof of Theorem 4.1 for the ETDRK2 scheme

The function $W_e(s)=U_e(t_m+s)$, $s\in[0,\tau]$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}W_e(s)}{\mathrm{d}s}+L_hW_e(s)=\Big(1-\dfrac{s}{\tau}\Big)Af_\kappa(U_e(t_m))+\dfrac{s}{\tau}Af_\kappa(U_e(t_{m+1}))+\widetilde{R}_m^{(2)}(s), \quad s\in(0,\tau),\\ W_e(0)=U_e(t_m), \end{cases} \tag{4.6}$$

where the truncated error $\widetilde{R}_m^{(2)}(s)$ satisfies

$$\max_{0\le m\le M_t-1}\sup_{s\in(0,\tau)}\|\widetilde{R}_m^{(2)}(s)\| \le \widetilde{C}_2(\tau^2+h^2)$$

with $\widetilde{C}_2$ depending on u_e, ε, κ, T and Ω. Let $e(s):=W(s)-W_e(s)$ with $W(s)$ determined by (2.10). The difference between (4.6) and (2.10) yields

$$\begin{cases} \dfrac{\mathrm{d}e(s)}{\mathrm{d}s}+L_he(s)=\Big(1-\dfrac{s}{\tau}\Big)[Af_\kappa(U^m)-Af_\kappa(U_e(t_m))]\\ \qquad\qquad +\dfrac{s}{\tau}[Af_\kappa(\widetilde{U}^{m+1})-Af_\kappa(U_e(t_{m+1}))]-\widetilde{R}_m^{(2)}(s), \quad s\in(0,\tau),\\ e(0)=U^m-U_e(t_m)=:e^m\in\mathcal{M}_h^0, \end{cases}$$

which gives the solution $e^{m+1}:=e(\tau)$ as

$$\begin{aligned} e^{m+1}=&\,\phi_{-1}(\tau L_h)e^m+\tau(\phi_0-\phi_1)(\tau L_h)[Af_\kappa(U^m)-Af_\kappa(U_e(t_m))]\\ &+\tau\phi_1(\tau L_h)[Af_\kappa(\widetilde{U}^{m+1})-Af_\kappa(U_e(t_{m+1}))]-\int_0^\tau \mathrm{e}^{-(\tau-s)L_h}\widetilde{R}_m^{(2)}(s)\mathrm{d}s. \end{aligned} \tag{4.7}$$

Acting $I+\tau L_h$ on both sides of (4.7) and taking the discrete L^2 inner product of the resulted equation with e^{m+1} yield

$$\|e^{m+1}\|^2+\varepsilon^2\tau\|Ae^{m+1}\|^2-\kappa\tau\langle Ae^{m+1},e^{m+1}\rangle=\text{RHS},\tag{4.8}$$

where

$$\begin{aligned}\text{RHS}=&\langle (I+\tau L_h)\phi_{-1}(\tau L_h)e^m,e^{m+1}\rangle\\&+\tau\langle (I+\tau L_h)(\phi_0-\phi_1)(\tau L_h)[f_\kappa(U^m)-f_\kappa(U_e(t_m))],Ae^{m+1}\rangle\\&+\tau\langle (I+\tau L_h)\phi_1(\tau L_h)[f_\kappa(\widetilde{U}^{m+1})-f_\kappa(U_e(t_{m+1}))],Ae^{m+1}\rangle\\&-\int_0^\tau\langle (I+\tau L_h)\mathrm{e}^{-(\tau-s)L_h}\widetilde{R}_m^{(2)}(s),e^{m+1}\rangle\mathrm{d}s.\end{aligned}$$

According to the assumption (3.2) and noting that $\|U_e(t_m)\|_\infty\le B$, we derive

$$\|f_\kappa(U^m)-f_\kappa(U_e(t_m))\|\le|3B^2-\kappa-1|\|e^m\|.$$

Since $\widetilde{U}^{m+1}$ is given by the ETD1 solution, we obtain from (3.2) and Theorem 3.1 that $\|\widetilde{U}^{m+1}\|\le B$, and thus, we have

$$\|f_\kappa(\widetilde{U}^{m+1})-f_\kappa(U_e(t_{m+1}))\|\le|3B^2-\kappa-1|\|\widetilde{U}^{m+1}-U_e(t_{m+1})\|.$$

Note that the estimate (4.2) implies

$$\|\widetilde{U}^{m+1}-U_e(t_{m+1})\|^2\le(2+4K\tau)\|e^m\|^2+4\tau^2\sup_{t\in(0,\tau)}\|R_m^{(1)}(t)\|^2.$$

According to Lemma 2.3 and noting $\frac{(3B^2-\kappa-1)^2}{\varepsilon^2}=K$, we derive

$$\begin{aligned}\text{RHS}\le&|||(I+\tau L_h)\phi_{-1}(\tau L_h)|||\|e^m\|\|e^{m+1}\|\\&+\tau|||(I+\tau L_h)(\phi_0-\phi_1)(\tau L_h)|||\|f_\kappa(U^m)-f_\kappa(U_e(t_m))\|\|Ae^{m+1}\|\\&+\tau|||(I+\tau L_h)\phi_1(\tau L_h)|||\|f_\kappa(\widetilde{U}^{m+1})-f_\kappa(U_e(t_{m+1}))\|\|Ae^{m+1}\|\\&+\int_0^\tau|||(I+\tau L_h)\mathrm{e}^{-(\tau-s)L_h}|||\mathrm{d}s\sup_{t\in(0,\tau)}\|\widetilde{R}_m^{(2)}(t)\|\|e^{m+1}\|\\\le&\|e^m\|\|e^{m+1}\|+\tau|3B^2-\kappa-1|\|e^m\|\|Ae^{m+1}\|\\&+\tau|3B^2-\kappa-1|\|\widetilde{U}^{m+1}-U_e(t_{m+1})\|\|Ae^{m+1}\|+\tau\sup_{t\in(0,\tau)}\|\widetilde{R}_m^{(2)}(t)\|\|e^{m+1}\|\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\|e^m\|^2+\frac{1}{2}\|e^{m+1}\|^2+\frac{(3B^2-\kappa-1)^2}{2\varepsilon^2}\tau\|e^m\|^2+\frac{\varepsilon^2}{2}\tau\|Ae^{m+1}\|^2\\
&\quad+\frac{(3B^2-\kappa-1)^2}{2\varepsilon^2}\tau\Big((2+4K\tau)\|e^m\|^2+4\tau^2\sup_{t\in(0,\tau)}\|R_m^{(1)}(t)\|^2\Big)+\frac{\varepsilon^2}{2}\tau\|Ae^{m+1}\|^2\\
&\quad+\frac{1}{2}\tau\sup_{t\in(0,\tau)}\|\widetilde{R}_m^{(2)}(t)\|^2+\frac{1}{2}\tau\|e^{m+1}\|^2\\
&=\frac{1}{2}\|e^m\|^2+\frac{1}{2}\|e^{m+1}\|^2+\varepsilon^2\tau\|Ae^{m+1}\|^2+\Big(\frac{3K}{2}+2K^2\tau\Big)\tau\|e^m\|^2+\frac{1}{2}\tau\|e^{m+1}\|^2\\
&\quad+\tau\Big(2K\tau^2\sup_{t\in(0,\tau)}\|R_m^{(1)}(t)\|^2+\frac{1}{2}\sup_{t\in(0,\tau)}\|\widetilde{R}_m^{(2)}(t)\|^2\Big).
\end{aligned}
$$

Combining with (4.8), we obtain

$$
\begin{aligned}
\frac{1}{2}\|e^{m+1}\|^2-\frac{1}{2}\|e^m\|^2\le\Big(\frac{3K}{2}+2K^2\tau\Big)\tau\|e^m\|^2+\frac{1}{2}\tau\|e^{m+1}\|^2&\\
+\tau\Big(2K\tau^2\sup_{t\in(0,\tau)}\|R_m^{(1)}(t)\|^2+\frac{1}{2}\sup_{t\in(0,\tau)}\|\widetilde{R}_m^{(2)}(t)\|^2\Big)&.
\end{aligned}
$$

Summing the above inequality from 0 to m and noting $e^0=0$ lead to

$$
\begin{aligned}
\frac{1}{2}\|e^{m+1}\|^2\le\Big(\frac{1}{2}+\frac{3K}{2}+2K^2\tau\Big)\tau\sum_{k=1}^{m}\|e^k\|^2+\frac{1}{2}\tau\|e^{m+1}\|^2&\\
+2K\tau\sum_{k=0}^{m}\tau^2\sup_{t\in(0,\tau)}\|R_m^{(1)}(t)\|^2+\frac{1}{2}\tau\sum_{k=0}^{m}\sup_{t\in(0,\tau)}\|\widetilde{R}_m^{(2)}(t)\|^2&,
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
(1-\tau)\|e^{m+1}\|^2\le(1+3K+4K^2\tau)\tau\sum_{k=1}^{m}\|e^k\|^2&\\
+4K\tau\sum_{k=0}^{m}\tau^2\sup_{t\in(0,\tau)}\|R_m^{(1)}(t)\|^2+\tau\sum_{k=0}^{m}\sup_{t\in(0,\tau)}\|\widetilde{R}_m^{(2)}(t)\|^2&,
\end{aligned}
$$

When $\tau\le\frac{1}{2}$, we have

$$
\begin{aligned}
\|e^{m+1}\|^2&\le(2+6K+8K^2)\tau\sum_{k=1}^{m}\|e^k\|^2+8K\tau\sum_{k=0}^{m}\tau^2\sup_{t\in(0,\tau)}\|R_m^{(1)}(t)\|^2+2\tau\sum_{k=0}^{m}\sup_{t\in(0,\tau)}\|\widetilde{R}_m^{(2)}(t)\|^2\\
&\le(2+6K+8K^2)\tau\sum_{k=1}^{m}\|e^k\|^2+(8KTC_1^2+2T\widetilde{C}_2^2)(\tau^2+h^2)^2.
\end{aligned}
$$

Using the discrete Gronwall's inequality, we obtain

$$
\|e^{m+1}\|^2\le(8KTC_1^2+2T\widetilde{C}_2^2)\mathrm{e}^{(2+6K+8K^2)T}(\tau^2+h^2)^2,
$$

that is,

$$\|e^{m+1}\| \leq \sqrt{8KTC_1^2+2T\widetilde{C}_2^2}\,\mathrm{e}^{(1+3K+4K^2)T}(\tau^2+h^2).$$

Then, we use the discrete inverse inequality (2.11) to obtain

$$\|e^{m+1}\|_\infty \leq \frac{C_{\mathrm{inv}}}{h}\|e^{m+1}\| \leq \sqrt{8KTC_1^2+2T\widetilde{C}_2^2}\,C_{\mathrm{inv}}\mathrm{e}^{(1+3K+4K^2)T}\Big(\frac{\tau^2}{h}+h\Big).$$

If $\tau \leq C'h$ and $h \leq \big[\sqrt{8KTC_1^2+2T\widetilde{C}_2^2}\,C_{\mathrm{inv}}\mathrm{e}^{(1+3K+4K^2)T}(C'^2+1)\big]^{-1}$, we obtain $\|e^{m+1}\|_\infty \leq 1$. Then, the L^∞ bound of the numerical solution U^{m+1} becomes

$$\|U^{m+1}\|_\infty \leq \|U^{m+1}-U_e(t_{m+1})\|_\infty + \|U_e(t_{m+1})\|_\infty \leq 1+\|U_e(t_{m+1})\|_\infty \leq B.$$

The proof is complete for the part of the Runge-Kutta scheme.

5 Conclusion

In this work, we provide a detailed convergence analysis of the exponential time differencing schemes for the Cahn-Hilliard equation in 2D and 3D cases. Taking the view that the numerical solutions are the exact solution plus small perturbations, we also give a theoretical justification on the uniform L^∞ boundedness of the numerical solutions, which is necessary to prove the error estimates. The uniform L^∞ boundedness is also desirable for numerical analysis of many other popular numerical schemes for phase field models to prove the discrete energy law. The techniques developed in this paper provide a framework which could be easily applied to those numerical methods to obtain the error estimates and the uniform L^∞ boundedness of the corresponding numerical solutions.

Acknowledgments

This work is supported by US National Science Foundation grant DMS-1818438 and National Natural Science Foundation of China grants 11801024 and 11871454.

References

[1] M. Ainsworth and Z.P. Mao, Analysis and approximation of a fractional Cahn-Hilliard equation, SIAM J. Numer. Anal., 55 (2017), 1689–1718.
[2] D.M. Anderson, G.B. McFadden and A.A. Wheeler, Diffuse-interface methods in fluid mechanics, Annu. Rev. Fluid Mech., 30 (1998), 139–165.
[3] A.L. Bertozzi, S. Esedoḡlu and A. Gillette, Inpainting of binary images using the Cahn-Hilliard equation, IEEE Trans. Image Process., 16 (2007), 285–291.
[4] L.A. Caffarelli and N.E. Muler, An L^∞ bound for solutions of the Cahn-Hilliard equation, Arch. Rational Mech. Anal., 133 (1995), 129–144.

[5] J.W. Cahn and J.E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, J. Chem. Phys., 28 (1958), 258–267.
[6] L.Q. Chen, Phase-field models for microstructure evolution, Annu. Rev. Mater. Res., 32 (2002), 113–140.
[7] S.M. Cox and P.C. Matthews, Exponential time differencing for stiff systems, J. Comput. Phys., 176 (2002), 430–455.
[8] Q. Du, L. Ju, X. Li and Z.H. Qiao, Stabilized linear semi-implicit schemes for the nonlocal Cahn-Hilliard equation, J. Comput. Phys., 363 (2018), 39–54.
[9] Q. Du, L. Ju, X. Li and Z.H. Qiao, Maximum principle preserving exponential time differencing schemes for the nonlocal Allen-Cahn equation, SIAM J. Numer. Anal., 57 (2019), 875–898.
[10] Q. Du, C. Liu and X.Q. Wang, Simulating the deformation of vesicle membranes under elastic bending energy in three dimensions, J. Comput. Phys., 212 (2006), 757–777.
[11] Q. Du and R. A. Nicolaides, Numerical analysis of a continuum model of phase transition, SIAM J. Numer. Anal., 28 (1991), 1310–1322.
[12] C.M. Elliott, The Cahn-Hilliard model for the kinetics of phase separation, In Mathematical models for phase change problems (Óbidos, 1988), 35–73, Internat. Ser. Numer. Math., 88, Birkhäuser, Basel, 1989.
[13] C.M. Elliott and D. A. French, Numerical studies of the Cahn-Hilliard equation for phase separation, IMA J. Appl. Math., 38 (1987), 97–128.
[14] D.J. Eyre, Unconditionally gradient stable time marching the Cahn-Hilliard equation, In Computational and Mathematical Models of Microstructural Evolution (San Francisco, CA, 1998), 39–46, Mater. Res. Soc. Sympos. Proc., 529, MRS, Warrendale, PA, 1998.
[15] X.B. Feng and A. Prohl, Error analysis of a mixed finite element method for the Cahn-Hilliard equation, Numer. Math., 99 (2004), 47–84.
[16] X.L. Feng, T. Tang and J. Yang, Stabilized Crank-Nicolson/Adams-Bashforth schemes for phase field models, East Asian J. Appl. Math., 3 (2013), 59–80.
[17] D. Furihata, A stable and conservative finite difference scheme for the Cahn-Hilliard equation, Numer. Math., 87 (2001), 675–699.
[18] G. Strang, Accurate partial difference methods. II. Non-linear problems, Numer. Math., 6 (1964), 37–46.
[19] Z. Guan, J. Lowengrub and C. Wang, Convergence analysis for second-order accurate schemes for the periodic nonlocal Allen-Cahn and Cahn-Hilliard equations, Math. Methods Appl. Sci., 40 (2017), 6836–6863.
[20] Z. Guan, C. Wang and S.M. Wise, A convergent convex splitting scheme for the periodic nonlocal Cahn-Hilliard equation, Numer. Math., 128 (2014), 377–406.
[21] Y.N. He, Y.X. Liu and T. Tang, On large time-stepping methods for the Cahn-Hilliard equation, Appl. Numer. Math., 57 (2007), 616–628.
[22] N.J. Higham, Functions of Matrices: Theory and Computation, SIAM, Philadelphia, PA, 2008.
[23] M. Hochbruck and C. Lubich, On Krylov subspace approximations to the matrix exponential operator, SIAM J. Numer. Anal., 34 (1997), 1911–1925.
[24] M. Hochbruck and A. Ostermann, Explicit exponential Runge-Kutta methods for semilinear parabolic problems, SIAM J. Numer. Anal., 43 (2005), 1069–1090.
[25] M. Hochbruck and A. Ostermann, Exponential integrators, Acta Numer., 19 (2010), 209–286.
[26] L. Ju, X. Li, Z.H. Qiao and H. Zhang, Energy stability and error estimates of exponential time differencing schemes for the epitaxial growth model without slope selection, Math. Comp., 87 (2018), 1859–1885.

[27] L. Ju and Z. Wang, Exponential time differencing gauge method for incompressible viscous flows, Comm. Comput. Phys., 22(2017), 517–541.

[28] L. Ju, J. Zhang and Q. Du, Fast and accurate algorithms for simulating coarsening dynamics of Cahn-Hilliard equations, Comput. Mater. Sci., 108 (2015), 272–282.

[29] L. Ju, J. Zhang, L.Y. Zhu and Q. Du, Fast explicit integration factor methods for semilinear parabolic equations, J. Sci. Comput., 62 (2015), 431–455.

[30] D. Li and Z.H. Qiao, On second order semi-implicit Fourier spectral methods for 2D Cahn-Hilliard equations, J. Sci. Comput., 70 (2017), 301–341.

[31] D. Li and Z.H. Qiao, On the stabilization size of semi-implicit Fourier-spectral methods for 3D Cahn-Hilliard equations, Commun. Math. Sci., 15 (2017), 1489–1506.

[32] D. Li, Z.H. Qiao and T. Tang, Characterizing the stabilization size for semi-implicit Fourier-spectral method to phase field equations, SIAM J. Numer. Anal., 54 (2016), 1653–1681.

[33] C. Liu, J. Shen and X.F. Yang, Dynamics of defect motion in nematic liquid crystal flow: Modeling and numerical simulation, Commun. Comput. Phys., 2 (2007), 1184–1198.

[34] Z.H. Qiao, Z.R. Zhang and T. Tang, An adaptive time-stepping strategy for the molecular beam epitaxy models, SIAM J. Sci. Comput., 33 (2011), 1395–1414.

[35] J. Shen, C. Wang, X.M. Wang and S.M. Wise, Second-order convex splitting schemes for gradient flows with Ehrlich-Schwoebel type energy: application to thin film epitaxy, SIAM J. Numer. Anal., 50 (2012), 105–125.

[36] J. Shen and X.F. Yang, Energy stable schemes for Cahn-Hilliard phase-field model of two-phase incompressible flows, Chin. Ann. Math. Ser. B, 31 (2010), 743–758.

[37] J. Shen and X.F. Yang, Numerical approximations of Allen-Cahn and Cahn-Hilliard equations, Discrete Contin. Dyn. Syst., 28 (2010), 1669–1691.

[38] G. Tierra and F. Guillén-González, Numerical methods for solving the Cahn-Hilliard equation and its applicability to related energy-based models, Arch. Comput. Methods Eng., 22 (2015), 269–289.

[39] S.M. Wise, C. Wang and J.S. Lowengrub, An energy stable and convergent finite difference scheme for the phase field crystal equation, SIAM J. Numer. Anal., 47 (2009), 2269–2288.

[40] C.J. Xu and T. Tang, Stability analysis of large time-stepping methods for epitaxial growth models, SIAM J. Numer. Anal., 44 (2006), 1759–1779.

[41] Z.R. Zhang and Z.H. Qiao, An adaptive time-stepping strategy for the Cahn-Hilliard equation, Commun. Comput. Phys., 11 (2012), 1261–1278.

[42] L.Y. Zhu, L. Ju and W.D. Zhao, Fast high-order compact exponential time differencing Runge-Kutta methods for second-order semilinear parabolic equations, J. Sci. Comput., 67 (2016), 1043–1065.