Implementation of Finite Difference Weighted Compact Nonlinear Schemes with the Two-Stage Fourth-Order Accurate Temporal Discretization

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Abstract. In this paper, we present a new two-stage fourth-order finite difference weighted compact nonlinear scheme (WCNS) for hyperbolic conservation laws with special application to compressible Euler equations. To construct this algorithm, apart from the traditional WCNS for the spatial derivative, it was necessary to first construct a linear compact/explicit scheme utilizing time derivative of flux at midpoints, which, in turn, was solved by a generalized Riemann solver. Combining these two schemes, the fourth-order time accuracy was achieved using only the two-stage time-stepping technique. The final algorithm was numerically tested for various one-dimensional and two-dimensional cases. The results demonstrated that the proposed algorithm had an essentially similar performance as that based on the fourth-order Runge-Kutta method, while it required 25 percent less computational cost for one-dimensional cases, which is expected to decline further for multidimensional cases.

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Key words: Hyperbolic conservation laws, finite difference method, Lax-Wendroff type time discretization, WCNS.

1 Introduction

For a long time, the development of high-order numerical methods for hyperbolic conservation laws has been actively discussed. Among various high-order methods, the finite difference method (FDM), which can be constructed in a dimension-by-dimension manner for uniform/curvilinear grid, has been appreciated for its simplicity, effectiveness, and low computational cost [10].

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Considering FDM, many specific finite difference schemes have been proposed. It is worth mentioning examples such as the finite difference center/upwind compact scheme [9, 12], the finite-difference version of the weighted ENO (WENO) scheme [11, 19], and the weighted compact nonlinear scheme (WCNS) [7].

Recently, the WCNS is becoming more widely used as it has been proven to have important advantages, such as preserving freestream [5, 14], and satisfying geometric conservation law [6]. Typically, the WCNS is used jointly with the method of lines, which allows separating the spatial discretization from the time evolution. The spatial discretization of the WCNS includes the following steps: (1) nonlinear reconstruction of an arbitrary variable from a cell node to a cell edge, (2) evaluation of the numerical flux on a cell edge, and (3) calculation of high-order central finite difference of the numerical flux from a cell edge to a cell node. Once this discretization has been parsed, one may apply an appropriate ordinary differential equation solver to achieve higher order temporal accuracy. In this study we applied the high-order total variation diminishing (TVD) Runge-Kutta schemes for this purpose.

However, similar to most of methods using the Runge-Kutta time-stepping approach for time integration, generally, the WCNS will require using effective interpolation stencils. Furthermore, as noted in [17], there is an order barrier for TVD Runge-Kutta methods with positive coefficients: they cannot be higher than the fourth-order accuracy.

In fact, it is reasonable to construct multistage multiderivative algorithms for time integration [3]. Recently, the two-stage fourth-order time-accurate Lax-Wendroff (L-W) time solver was introduced, particularly for application with the hyperbolic conservation laws [13]. The two-stage L-W type time-stepping method is able to achieve fourth-order time accuracy solely owing to the use of both flux derivative and its time derivative. Previous works [4, 8, 15, 16] have already demonstrated that this solver has advantages in terms of its computational efficiency.

Therefore, it is beneficial to investigate how to implement this two-stage fourth-order Lax-Wendroff type time solver in the WCNS. To the author’s knowledge, there has been no previous research on this topic. In this work, we developed a new two-stage fourth-order WCNS with application of compressible Euler equations.

The rest of the paper is organized as follows. In Section 2, we provide a brief review of the two-stage fourth-order Lax-Wendroff time discretization method. In Section 3, we describe in detail the specific algorithms of the proposed approach. Extension to multidimensional nonlinear systems is also outlined in this section. Numerical results are provided in Section 4 and conclusions are drawn in Section 5.

2 Two-stage fourth-order time-accurate solver

In this section, we briefly review the two-stage fourth-order Lax-Wendroff type time discretization method [13]. Let us consider the following time-dependent nonlinear hyper-
bolic conservation laws
\[
\frac{\partial u}{\partial t} = \mathcal{R}(u),
\]  
(2.1)

subject to the initial data at time \( t = t_n \),
\[
u(t)|_{t = t_n} = u^n,
\]  
(2.2)

where \( \mathcal{R} \) is an operator for spatial derivative. The solution at the next time step \( t_{n+1} \) can be obtained as follows:
\[
u^* = u^n + \frac{1}{2} \Delta t \mathcal{R}(u^n) + \frac{1}{8} \Delta t^2 \frac{\partial}{\partial t} \mathcal{R}(u^n),
\]  
(2.3)

\[
u^{n+1} = u^n + \Delta t \mathcal{R}(u^n) + \frac{1}{6} \Delta t^2 \left( \frac{\partial}{\partial t} \mathcal{R}(u^n) + 2 \frac{\partial}{\partial t} \mathcal{R}(u^*) \right).
\]  
(2.4)

It has been proved that [13] this Lax-Wendroff type time discretization method has fourth-order temporal accuracy. The advantage of the two-stage fourth-order accurate temporal discretization is that it only requires two-stage time-stepping to achieve the expected fourth-order accuracy. It was discovered that the computational cost can be considerably reduced compared with that of the multistage Runge-Kutta method of the same order [4]. To see the details of this method, refer to [13].

3 Two-stage fourth-order WCNS

3.1 One-dimensional case

To define the discretization of one-dimensional time-dependent nonlinear hyperbolic conservation laws, the following can be written
\[
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0,
\]  
(3.1)

considering a uniform grid defined by the points \( x_i = i \Delta x, i = 1, \ldots, N \), which are also called cell centers, with cell boundaries given by \( x_{i+1/2} = x_i + \frac{\Delta x}{2} \), where \( \Delta x \) is the uniform grid spacing. The spatial discretization can be obtained by implicitly defining the numerical flux function \( h(x, t) \) as follows
\[
f(u) = \frac{1}{\Delta x} \int_{x-\Delta x/2}^{x+\Delta x/2} h(\xi, t) d\xi.
\]  
(3.2)

Therefore, the spatial derivative in Eq.(3.1) is exactly approximated by a conservative finite difference formula at the cell boundaries,
\[
\frac{d}{dt} u_i(t) = \mathcal{R}(u_i) = -\frac{h_{i+1/2} - h_{i-1/2}}{\Delta x},
\]  
(3.3)
where $h_{i\pm 1/2} = h(x_{i\pm 1/2},t)$, and $u_i(t)$ is a numerical approximation to the point value $u(x_i,t)$ at a point $x_i$ [19]. Instead of using point values to approximate $h_{i+1/2}$ [19], the WCNS [7] use a linear midpoint value based scheme, namely, the fourth-order compact difference cell-centered scheme, with $a = \frac{3}{8}(3 - 2\alpha)$ and $b = \frac{1}{8}(22\alpha - 1)$ [12]
\[
\alpha h_{i+1/2} + h_{i+1/2} + \alpha h_{i+3/2} \\
\approx \frac{c}{5} f_{i-3/2} + \left(\frac{b}{3} + \frac{c}{5}\right) f_{i-1/2} + \left(a + \frac{b}{3} + \frac{c}{5}\right) f_{i+1/2} + \left(\frac{b}{3} + \frac{c}{5}\right) f_{i+3/2} + \frac{c}{5} f_{i+5/2}. \tag{3.4}
\]
In this scheme the expression of $h_{i+1/2}$ is linear explicitly (with $\alpha = 0$ in Eq. (3.4)) or compactly (with $\alpha \neq 0$ in Eq. (3.4)). The midpoint values $f_{i+1/2+m}$ ($m = -4, \ldots, 3$) can be obtained by direct flux-split WENO-type interpolations [23] or various Riemann solvers (provided nonlinear interpolation of an arbitrary variable, e.g., a conservative variable from a cell node to a cell edge) [7].

In addition to this classical knowledge, we need more detailed information on the time derivative. To be specific, to implement any conservative FDM in the two-stage fourth-order time solver (Eqs. (2.3) and (2.4)), we need to develop a scheme to calculate the time derivative of the spatial derivative, i.e.,
\[
\frac{d}{dt} \left(\frac{d}{dt} u_i(t)\right) = \frac{\partial}{\partial t} n_i = -\frac{h_{i+1/2} - h_{i-1/2}}{\Delta x}. \tag{3.5}
\]
Using this scheme and Eq. (3.3), we can reformulate Eqs. (2.3) and (2.4) as follows
\[
u_i^n = u_i^n - \frac{\Delta t}{2\Delta x} \left(h_{i+1/2}^n - h_{i-1/2}^n\right), \tag{3.6}
\]
\[
u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \left(h_{i+1/2}^{4th} - h_{i-1/2}^{4th}\right), \tag{3.7}
\]
in which
\[
h_{i\pm 1/2}^* = h_{i\pm 1/2} + \frac{1}{4} \Delta t h_{i\pm 1/2}^{'}, \tag{3.8}
\]
\[
h_{i\pm 1/2}^{4th} = h_{i\pm 1/2} + \frac{1}{6} \Delta t \left(h_{i\pm 1/2}^{'(2)} \right)^*, \tag{3.9}
\]
where $(h_{i\pm 1/2}^{'(2)})^*$ is the associated numerical flux at $x_{i\pm 1/2}$ according to the intermediate state $u^*$.

In this paper, because a sixth-order explicit scheme ($\alpha = 0, c = \frac{3}{640}$ in Eq. (3.4)) is used to provide the value of $h_{i+1/2}$, we propose that $h_{i+1/2}^{'(2)}$ is calculated accordingly as per the formula
\[
h_{i+1/2}^{'(2)} \approx \frac{3}{640} \left(\frac{\partial f}{\partial t}\right)_{i-3/2} - \frac{29}{480} \left(\frac{\partial f}{\partial t}\right)_{i-1/2} + \frac{1067}{960} \left(\frac{\partial f}{\partial t}\right)_{i+1/2} - \frac{29}{480} \left(\frac{\partial f}{\partial t}\right)_{i+3/2} + \frac{3}{640} \left(\frac{\partial f}{\partial t}\right)_{i+5/2}. \tag{3.10}
\]
At this point, the question of how to calculate $\frac{\partial f}{\partial t}$ at midpoints has emerged. In this paper, we propose solving this problem by applying a Generalized Riemann problem solver [22], such as the truly nonlinear GRP solver [1] or the arbitrary high order Godunov approach (ADER) [20].

As a result, the architecture of the two-stage fourth-order WCNS algorithm is built. The details of the proposed algorithm can be summarized as follows.

**Step 1.** Interpolate the polynomials $u^L(x)$, $u^R(x)$ in cell $[x_{i-1/2}, x_{i+1/2}]$. To achieve non-oscillation in the results, the WENO interpolation methodology [7] can be used. In this study, we employed an approach of construing the polynomials $u^L(x)$ that can be described as follows.

At each fixed $x_{i+1/2}$, the average state $u^A_{i+1/2}$ is computed by the simple mean or the Roe mean. Let us denote as $\lambda_p$, $l_p$ and $r_p$ the $p$th eigenvalue, left eigenvector, and right eigenvector of the coefficient matrix $A = \frac{df}{du} |_{u = u^A_{i+1/2}}$. The interpolated polynomial in a cell $[x_{i-1/2}, x_{i+1/2}]$ for the $p$th characteristic variables

$$q_{i,p} = l_p \cdot u,$$

where $\omega_k$ and $\overline{q}_{i,p}^{(k)}(x)$ ($k = 1,\cdots, 3$) are the nonlinear weight and interpolated polynomial, respectively, according to the $k$th stencil $[x_{i-3+k}, x_{i-2+k}, x_{i-1+k}]$, and nonlinear weights of three stencils in question are obtained by the improved WENO methodology [2] with the optimal weights $C_1 = \frac{1}{6}, C_2 = \frac{10}{19}, C_3 = \frac{5}{19}$ [7].

Thereafter, after transforming back into the physical space, we can obtain the interpolated polynomials $u_i(x)$ in a cell $[x_{i-1/2}, x_{i+1/2}]$. Consequently, we can obtain $u^L_{i+1/2}$, $u^R_{i+1/2}$, $u^A_{i+1/2}$, and $\overline{q}_{i+1/2}^{(k)}$. We can obtain $u_{i+1/2}$ by solving the classical Riemann problem with two constant states $u^L_{i+1/2}$ and $u^R_{i+1/2}$. In this study, the Harten-Lax-van Leer-Contact solver (HLLC) [21] proposed by Toro et al. is used. We can obtain $\overline{q}_{i+1/2}^{(k)}$ by using the Generalized Riemann problem solver

$$\frac{\partial u}{\partial t} |_{i+1/2} = -A^+ (u_{i+1/2}) \left( \frac{\partial u}{\partial x} \right)^L_{i+1/2} - A^- (u_{i+1/2}) \left( \frac{\partial u}{\partial x} \right)^R_{i+1/2},$$

where $\Lambda = \text{diag} \{ \lambda_k \}$, i.e., is formed by the eigenvalues $\lambda_k$ of $A(u_{i+1/2})$, $A^\pm = RA^\pm L$, $R$ and $L$ is the right and left eigenmatrix of $A(u_{i+1/2})$, $\Lambda^+ = \text{diag} \{ \max(\lambda_k, 0) \}$, $\Lambda^- = \text{diag} \{ \min(\lambda_k, 0) \}$.
Step 3. Calculate \( f_{i+1/2} \) and \( \left( \frac{df}{dt} \right)_{i+1/2} \). It is possible to calculate \( f_{i+1/2} \) by using various classical Riemann solvers, for example, the HLLC solver as used in this study. However, the question is how to calculate \( \left( \frac{df}{dt} \right)_{i+1/2} \). In this study, the Cauchy–Kovalevskaya procedure,

\[
\left( \frac{df}{dt} \right)_{i+1/2} = \left( \frac{df}{\partial u} \right) \left( \frac{\partial u}{\partial t} \right)_{i+1/2}
\]

is used.

Step 4. Finally, we can obtain \( h_{i+1/2} \) and \( h'_{i+1/2} \) by the approximation in Eq. (3.3) and Eq. (3.10). This step completes the algorithm as we can perform the time update now.

3.2 Two-dimensional case

Although the FDM is usually constructed in a dimension-by-dimension manner for uniform/curvilinear grid, it is worth discussing in more detail the available ways to extend the proposed algorithm for the multidimensional case.

For the two-dimensional case, the computational domain \( \Omega \) is divided into uniform rectangular meshes \( K_{ij}, \Omega = \bigcup_{i\in I,j\in J} K_{ij} \), where \( K_{ij} = (x_{i-1/2},x_{i+1/2}) \times (y_{j-1/2},y_{j+1/2}) \) with \( (x_i,y_j) \) as the center, \( \Delta x \) and \( \Delta y \) is the uniform \( x \)-direction and \( y \)-direction grid spacing respectively.

Specific steps are the same as for the one-dimensional case. A step using the dimension-by-dimension technology with Eq. (3.13) is replaced by the two-dimensional solver [13] (for the \( \left( \frac{\partial u}{\partial t} \right)_{ij+1/2} \) is the same, and omit for brevity)

\[
\left( \frac{\partial u}{\partial t} \right)_{i+1/2,j} = -A^+(u_{i+1/2,j}) \left( \frac{\partial u}{\partial x} \right)_{i+1/2,j} + RI^+ \left( \frac{\partial g(u^L_{i+1/2,j})}{\partial y} \right)_{i+1/2,j} - \frac{RI^+}{2} \frac{\partial g(u^L_{i+1/2,j})}{\partial y} \nonumber
\]

\[
\left( \frac{\partial u}{\partial t} \right)_{i+1/2,j} = -A^-(u_{i+1/2,j}) \left( \frac{\partial u}{\partial x} \right)_{i+1/2,j} - RI^- \left( \frac{\partial g(u^R_{i+1/2,j})}{\partial y} \right)_{i+1/2,j} + \frac{RI^-}{2} \frac{\partial g(u^R_{i+1/2,j})}{\partial y} \nonumber
\]

where \( \Lambda = diag \{ \lambda_k \} \), i.e., is formed by the eigenvalues \( \lambda_k \) of \( A(u_{i+1/2,j}) \), \( A^+ = RA^+L \), \( R \) and \( L \) is the right and left eigenmatrix of \( A(u_{i+1/2,j}) \), \( \Lambda^+ = diag \{ \max(\lambda_k,0) \} \), \( \Lambda^- = diag \{ \min(\lambda_k,0) \} \), \( I^+ = \frac{1}{2} diag \{ 1 + sign(\lambda_k) \} \), \( I^- = \frac{1}{2} diag \{ 1 - sign(\lambda_k) \} \), \( g \) the flux along \( y \) direction (multidimensional effect). In this solver, we need also to calculate \( \frac{\partial g(u^L_{i+1/2,j})}{\partial y} \) and \( \frac{\partial g(u^R_{i+1/2,j})}{\partial y} \). Since tangent derivative values of \( u^L_{i+1/2,j} \) and \( u^R_{i+1/2,j} \) are available for each \( j \), it is possible to use various finite difference methods, for example, the sixth-order difference scheme as used in this study.
4 Numerical tests

To estimate the feasibility of the proposed algorithm, several typical numerical tests for one- and two-dimensional cases were performed. In the rest of this paper, “LW4+WCNS-E-5” refers to the proposed algorithm described above, and “RK4+WCNS-E-5” refers to the fifth-order WCNS-E-5 scheme [7] combined with the forth-order Runge-Kutta method [18]. For the computations described in this subsection, the Courant-Friedrichs-Lewy condition (CFL) value is set to 0.4.

4.1 One-dimensional Riemann problems

Firstly, let us consider two one-dimensional Riemann problems: the Sod problem, and the 123 problem. For the first problem, the initial condition is defined as follows

\[(\rho, u, p) = \begin{cases} (1,0,1), & \text{if } 0 \leq x \leq 0.5, \\ (0.125,0,0.1), & \text{if } 0.5 \leq x \leq 1, \end{cases}\]

\[N = 101 \text{ is used, and the final time is } t = 0.2.\]

For the second problem, the initial condition is written as

\[(\rho, u, p) = \begin{cases} (1,-2,0.4), & \text{if } 0 \leq x \leq 0.5, \\ (1,2,0.4), & \text{if } 0.5 \leq x \leq 1, \end{cases}\]

\[N = 101 \text{ is used, and the final time is } t = 0.1.\]

Fig. 1 shows the density, velocity, and pressure computed according to two methods under consideration. From Fig. 1, it can be seen that both symbols are located closely to the referred line and two symbols are nearly overlapped in the two numerical cases. The results show that appropriate conformance with the reference solution is obtained for both “LW4+WCNS-E-5” and “RK4+WCNS-E-5”. Furthermore, “LW4+WCNS-E-5” produces essentially the same results as “RK4+WCNS-E-5”, which demonstrates the feasibility of the proposed algorithms.

4.2 Shock density-wave interaction problem

The initial conditions are set by a Mach 3 shock interacting with a perturbed density field

\[(\rho, u, p) = \begin{cases} (3.857,2.629,10.333), & \text{if } 0 \leq x \leq 1, \\ (1+0.2\sin(5x),0,1), & \text{if } 1 \leq x \leq 10, \end{cases}\]

and the final time is \(t = 1.8\). A zero-gradient boundary condition is applied at \(x = 0\) and \(x = 10\). Moreover, we denote the solution provided by “RK4+WCNS-E-5” on a grid of 5001 points as the reference solution.

Fig. 2 and Fig. 3 show density profiles on the grids of 201 and 401 points, respectively. In Fig. 2 we can see that the grid resolution is too low for two methods to resolve the
Figure 1: One-dimensional Riemann problem: (left) Sod problem; (right) 123 problem.
high frequency waves. The resolution of “RK4+WCNS-E-5” method is slightly higher than that of “LW4+WCNS-E-5” method, while the difference between the results of two methods is small. In Fig. 3 we can see that two methods produce the reference resolution on the grid of 401 points, and the results of two methods are nearly similar to each other. This demonstrates the feasibility of the proposed algorithm in case of the multifrequency wave problem.
4.3 Two-dimensional Riemann problems

In this section, we consider two examples for two-dimensional Riemann problems. The first one is the interaction of shocks, rarefaction waves and vortex sheets [13]. The computational domain is \([0,1] \times [0,1]\), and the initial conditions are

\[
\begin{align*}
(\rho, u, v, p) &= \begin{cases} 
(1,0,0.3,1), & 0 \leq x \leq 1, 0.5 \leq y \leq 1, \\
(2,0,-0.3,1), & 0 \leq x \leq 0.5, 0.5 \leq y \leq 1, \\
(1.0625, 0, 0.2145, 0.4), & 0 \leq x \leq 0.5, 0 \leq y \leq 0.5, \\
(0.5179, 0, -0.4259, 0.4), & 0.5 \leq x \leq 1, 0 \leq y \leq 0.5.
\end{cases}
\end{align*}
\]

The computation is performed on the uniform grid of 201 \times 201 points, where first-order extrapolation boundary conditions are applied on both left and right sides of the domain. Fig. 4 shows density contours of this problem at time \(t = 0.3\). In this figure we can see that “LW4+WCNS-E-5” and “RK4+WCNS-E-5” still produce essentially similar results.

The second example corresponds to the interaction of rarefaction waves and vortex sheets [13]. The computational domain is \([0,1] \times [0,1]\), and the initial conditions are

\[
\begin{align*}
(\rho, u, v, p) &= \begin{cases} 
(1,0.1,0.1,1), & 0.5 \leq x \leq 1, 0.5 \leq y \leq 1, \\
(0.5197, -0.6259, 0.1, 0.4), & 0 \leq x \leq 0.5, 0.5 \leq y \leq 1, \\
(0.8, 0.1, 0.1, 0.4), & 0 \leq x \leq 0.5, 0 \leq y \leq 0.5, \\
(0.5179, 0.1, -0.6259, 0.4), & 0.5 \leq x \leq 1, 0 \leq y \leq 0.5.
\end{cases}
\end{align*}
\]

The computation is performed on the uniform grid of 501 \times 501 points, where first-order extrapolation boundary conditions are applied on both the left and right sides of

Figure 4: Interaction of shocks, rarefaction waves and vortex sheets problem on a grid of 201 \times 201 points: density profiles obtained using (left) LW4 and (right) RK4.
the domain. Fig. 5 shows density contours of this problem at time \( t = 0.055 \). In these figures we can see that “LW4+WCNS-E-5” and “RK4+WCNS-E-5” still produce essentially similar results. This result further confirms the conclusion stated above.

4.4 Richtmyer-Meshkov instability

We use a rectangular domain \([0,8] \times [0,0.5]\) with a shock Mach number \( M_s = 3 \) interacting with a single mode sinusoidal perturbed interface along xenon (Xe) and argon (Ar) gases (the specific heat ratio \( \gamma = \frac{5}{3} \)). The initial shape of the perturbed interface was generated with

\[
x_{\text{dis}} = 0.2 + 0.1\cos(2\pi(y - 0.5)), \quad 0 < y < 0.5
\]

and the initial condition of the single mode Richtmyer-Meshkov instability is specified as follows:

\[
(\rho, u, v, p) = \begin{cases} 
(3, 2.582, 0, 11), & 0 < x < 0.1, \\
(1, 0, 0, 1), & 0.1 < x < x_{\text{dis}}, \\
(0.1, 0, 0, 1), & x_{\text{dis}} \leq x \leq 8.
\end{cases}
\]

For the bottom and top boundaries, reflective boundary conditions are used. Non-reflective boundary conditions are applied for the left and right boundaries. The algorithm is run until \( t = 1.5 \), and the results for \([5,7] \times [0,0.5]\) are provided in Fig. 6. In these figures we can see that “LW4+WCNS-E-5” still produces a feasible result.
4.5 Computational cost

To test the efficiency of the proposed algorithm, the computational cost of "LW4+WCNS-E-5" (further referred as "LW4") and "RK4+WCNS-E-5" (further referred as "RK4") are compared with each other.

For one-dimensional cases, the Sod problem and the Shu-Osher problem, the settings of which are described in Section 4.1 and Section 4.2 respectively, are computed on a grid of 1001 points. The CFL number changed from 0.1 to 1.1.

Fig. 7 shows the computational cost based on the "LW4" and "RK4" schemes for the Sod problem and Shu-Osher problem. In the figure we can see that the computational cost is always more expensive for "RK4" scheme than the one for "LW4" scheme. For both test cases, the cost of "RK4" scheme is approximately 25 percent greater than that of "LW4" scheme.

For two-dimensional cases, the computational cost is not deterministic as there is no unified approach to calculate tangent derivatives, namely, \( \frac{\partial g(u_{L_{i+1/2}})}{\partial y} \) and \( \frac{\partial g(u_{R_{i+1/2}})}{\partial y} \). For the linear scheme used in this study, numerical experiments demonstrate that nearly 55 percent CPU time can be saved using the proposed scheme compared to the "RK4" type schemes of the same order. If nonlinear scheme is used, the computational cost is expected to be larger, but still less than the one of the "RK4" type schemes of the same order.
To summarize, the results demonstrate the efficiency of the present algorithm. This advantage can be expected to be more evident for higher dimensional problems, as in the proposed algorithm the efficiency is achieved with the less reconstructions required, in comparison to “RK4” type schemes. The computational efficiency is crucial to achieve in the simulations of multiscale problems, such as turbulent mixing.

5 Conclusions

In this study, we investigated the implementation of a two-stage fourth-order time discretization solver into finite difference WCNS. The main difficulty in implementation was the fact that the spatial derivative and the time derivative of the spatial derivative are required simultaneously. Therefore, we developed the linear compact/explicit scheme combined with time derivative flux at midpoints, which, in turn, was solved by the generalized Riemann solver. Integrated with the traditional WCNS for the spatial derivative, a novel two-stage fourth-order WCNS algorithm was proposed. The key advantage of the proposed method is that it only requires two-stage time-stepping to achieve the fourth-order accuracy, in comparison to five stages required for the TVD Runge-Kutta method of the same order. The computational cost of the proposed algorithm is less than that of the Runge-Kutta method and is expected to be diminished further for a multidimensional problem. This advantage is of great significance in massively parallel simulations for multi-scale problems, such as turbulent mixing.

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