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A Time-Splitting Spectral Method for Three-Wave Interactions in Media with Competing Quadratic and Cubic Nonlinearities

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Abstract. This paper introduces an extension of the time-splitting spectral (TSSP) method for solving a general model of three-wave optical interactions, which typically arises from nonlinear optics, when the transmission media has competing quadratic and cubic nonlinearities. The key idea is to formulate the terms related to quadratic and cubic nonlinearities into a Hermitian matrix in a proper way, which allows us to develop an explicit and unconditionally stable numerical method for the problem. Furthermore, the method is spectral accurate in transverse coordinates and second-order accurate in propagation direction, is time reversible and time transverse invariant, and conserves the total wave energy (or power or the norm of the solutions) in discretized level. Numerical examples are presented to demonstrate the efficiency and high resolution of the method. Finally the method is applied to study dynamics and interactions between three-wave solitons and continuous waves in media with competing quadratic and cubic nonlinearities in one dimension (1D) and 2D.

AMS subject classfications: 65M70, 78M25, 42A10

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1 Introduction

In nonlinear optics, based on the model for the type-II [11] second-harmonic-generating (SHG) systems and the one for the co-propagation of two orthogonal linear polarizations

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in the lossless Kerr medium [9], a general model of three-wave optical interactions system, which combines the quadratic and cubic nonlinearities and birefringence, can be formulated as follows [9,11,12]:

$$iu_z+\frac{1}{2}\Delta u+\alpha v^*w+\gamma_1\left(\frac{1}{4}|u|^2+\frac{1}{6}|v|^2+2|w|^2\right)u+\frac{\gamma_1}{12}v^2u^*+\gamma_2|w|^2v+bu=0, \tag{1.1}$$

$$iv_z+\frac{1}{2}\Delta v+\alpha u^*w+\gamma_1\left(\frac{1}{4}|v|^2+\frac{1}{6}|u|^2+2|w|^2\right)v+\frac{\gamma_1}{12}u^2v^*+\gamma_2|w|^2u-bv=0, \tag{1.2}$$

$$2iw_z+\frac{1}{2}\Delta w+\alpha uv+2\gamma_1\left(2|w|^2+|u|^2+|v|^2\right)w+\gamma_2(uv^*+u^*v)w-qw=0, \tag{1.3}$$

where $u=u(\mathbf{x},z)$, $v=v(\mathbf{x},z)$ and $w=w(\mathbf{x},z)$ are complex-valued functions, the fields u and v are complex envelopes of the two components with fundamental frequency, w is that of the single second harmonic component, and f^* is the complex conjugate of a function f. For the parameters in (1.1)-(1.3), b is a real birefringence coefficient, q is the phase-mismatch parameter that controls the SHG process, α is the coefficient for quadratic nonlinearity, and γ_1 and γ_2 are coefficients for cubic nonlinearities. All these parameters are real, and they may be positive or negative, with a constraint that γ_1 and γ_2 have the same sign. The cases $\gamma_1,\gamma_2>0$, and resp. $\gamma_1,\gamma_2<0$, correspond to the self-focusing and self-defocusing cubic nonlinearities.

The system (1.1)-(1.3) is written for the paraxial evolution in the *spatial domain*, so that $z\geq 0$ is the propagation distance, $\mathbf{x}\in\mathbb{R}^d$ ($d=1,2$) is the transverse coordinate in the corresponding planar waveguide, and the Δ operator accounts for transverse coordinates. Furthermore, it conserves a dynamical invariant (power, alias the norm of the solution, in the temporal domain, it would be wave energy), i.e.,

$$\begin{aligned} &E(z)\\ =&\int_{\mathbb{R}^d}\left[|u(\mathbf{x},z)|^2+|v(\mathbf{x},z)|^2+4|w(\mathbf{x},z)|^2\right]d\mathbf{x}\equiv\int_{\mathbb{R}^d}\left[|u(\mathbf{x},0)|^2+|v(\mathbf{x},0)|^2+4|w(\mathbf{x},0)|^2\right]d\mathbf{x}\\ =&\ E(0),\quad z\geq 0. \end{aligned} \tag{1.4}$$

In addition, the system also conserves the momentum and Hamiltonian, which will not be explicitly used in this work.

The general form of (1.1)-(1.3) covers many three-wave interactions in nonlinear optics. For example, when $\gamma_1=\gamma_2=0$, i.e. all the cubic nonlinear terms are dropped, it is reduced to the so-called type-II SHG system which is a subject that has been studied in detail theoretically and experimentally, see reviews [11,15]. In this case, the material birefringence is employed to phase-match two orthogonal linearly polarized components of the fundamental-frequency wave to a single second-harmonic component. In addition, if $b=0$ and $u=v$, it is further simplified to a two-wave setting, i.e. type-I SHG system without walk-off between harmonic waves [13]. The solutions of this simplified two-wave