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# On Singular Closures for the 5-Moment System in Kinetic Gas Theory

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**Abstract.** Moment equations provide a flexible framework for the approximation of the Boltzmann equation in kinetic gas theory. While moments up to second order are sufficient for the description of equilibrium processes, the inclusion of higher order moments, such as the heat flux vector, extends the validity of the Euler equations to non-equilibrium gas flows in a natural way.

Unfortunately, the classical closure theory proposed by Grad leads to moment equations, which suffer not only from a restricted hyperbolicity region but are also affected by non-physical sub-shocks in the continuous shock-structure problem if the shock velocity exceeds a critical value. A more recently suggested closure theory based on the maximum entropy principle yields symmetric hyperbolic moment equations. However, if moments higher than second order are included, the computational demand of this closure can be overwhelming. Additionally, it was shown for the 5-moment system that the closing flux becomes singular on a subset of moments including the equilibrium state.

Motivated by recent promising results of closed-form, singular closures based on the maximum entropy approach, we study regularized singular closures that become singular on a subset of moments when the regularizing terms are removed. In order to study some implications of singular closures, we use a recently proposed explicit closure for the 5-moment equations. We show that this closure theory results in a hyperbolic system that can mitigate the problem of sub-shocks independent of the shock wave velocity and handle strongly non-equilibrium gas flows.

**AMS subject classifications**: 35Q20, 35Q35

**Key words**: Non-equilibrium gas dynamics, hyperbolic moment equations, kinetic theory, maximum entropy closures.

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## 1 Introduction

Fluid equations supplied with the constitutive relations of Navier-Stokes and Fourier rely on the fundamental premiss of sufficiently weak deviations from local thermodynamic equilibrium. While this prerequisite is satisfied for many processes in gases in the collision-dominated regime, stronger deviations from local thermodynamic equilibrium can occur in moderately rarefied gas flows, rendering the constitutive relations unsound.

Moment equations are derived directly from the Boltzmann equation and offer the promise of a general framework for the accurate and efficient description of rarefied gases by the inclusion of an extended set of macroscopic fields. Moment systems are an attractive alternative to the particle based DSMC approach especially in the transition regime as they do not suffer from a severe time-step restriction or statistical noise [20].

Moment approximations of the Boltzmann equation yield hyperbolic systems of balance laws composed of a transport operator, a relaxation term describing the direct particle interactions and possibly further source terms, e.g. accounting for the effects of macroscopic force fields. An important feature of moment equations is the hyperbolicity of the transport operator as loss of hyperbolicity renders the system ill-posed and numerically unstable, see [25].

While the closure theory proposed by Grad in [9] yields closed form expressions for the closing fluxes, the resulting system suffers from a loss of hyperbolicity beyond small deviations from local equilibrium, limiting the usability of this closure to simulations of weakly non-equilibrium flows. Furthermore, this closure suffers from unphysical sub-shocks in the continuous shock-structure problem, see e.g. [10].

Further interest into the study of sub-shocks was sparked when Ruggeri [22] argued the non-existence of smooth $C^1$ solutions if the inflow Mach number exceeds a critical value, leading to a series of papers on the subject of sub-shocks [2, 3, 27]. In [6] Boillat and Ruggeri provided an answer to the question of when continuous solutions to moment systems can occur by proving the non-existence of $C^1$ solutions to the shock wave structure problem for generic balance laws endowed with a convex entropy if the shock velocity exceeds the largest characteristic velocity evaluated in the equilibrium state in front of the shock.

Several approaches to the sub-shock problem have been considered in the literature. As shown by Boillat and Ruggeri [5] in the framework of extended thermodynamics, the largest characteristic speed evaluated in the equilibrium state is bounded from below by a monotonically increasing function of the highest tensorial degree of the moment system. Thus one possible approach to find smooth solutions is to consider systems with a higher number of moment equations, see e.g. [3]. However, this approach turns out to be rather impractical as the required number of moment equations would have to be increased drastically even for moderate Mach numbers, see [2]. A completely different approach has been considered in [24], where a regularization of Grad's equations by the inclusion of parabolic terms was proposed. The resulting R13 equations allow smooth shock-structure solutions as shown in [26].