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Fast Solver for the Local Discontinuous Galerkin Discretization of the KdV Type Equations

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Abstract. In this paper, we will develop a fast iterative solver for the system of linear equations arising from the local discontinuous Galerkin (LDG) spatial discretization and additive Runge-Kutta (ARK) time marching method for the KdV type equations. Being implicit in time, the severe time step ($\Delta t = \mathcal{O}(\Delta x^k)$, with the k-th order of the partial differential equations (PDEs)) restriction for explicit methods will be removed. The equations at the implicit time level are linear and we demonstrate an efficient, practical multigrid (MG) method for solving the equations. In particular, we numerically show the optimal or sub-optimal complexity of the MG solver and a two-level local mode analysis is used to analyze the convergence behavior of the MG method. Numerical results for one-dimensional, two-dimensional and three-dimensional cases are given to illustrate the efficiency and capability of the LDG method coupled with the multigrid method for solving the KdV type equations.

AMS subject classifications: 65M60, 35Q53

Key words: KdV type equations, local discontinuous Galerkin methods, multigrid algorithm, additive Runge-Kutta methods, local mode analysis.

1 Introduction

In this paper, we apply the multigrid (MG) solver to solve the system of algebraic equations arising from the local discontinuous Galerkin (LDG) spatial discretization and additive Runge-Kutta (ARK) time marching method for the KdV type equations containing third derivatives terms

$$u_t + f(u)_x + u_{xxx} = 0, \tag{1.1}$$

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and the fifth-order KdV type equations

$$u_t+f(u)_x+g(u_x)_{xx}+u_{xxxxx}=0, \tag{1.2}$$

in $\Omega\in\mathbb{R}^d(d\le 3)$, where $f(u)\ge 0$ and $g(p)$ are arbitrary (smooth) functions.

The LDG methods for these two types of equations were derived by Yan and Shu [18, 19], which were high order accurate, stable and flexible for arbitrary h and p adaptivity. In these two papers, time discretization was by the explicit Runge-Kutta method with a suitably small Δt for stability ($\Delta t=\mathcal{O}(\Delta x^3)$ for Eq. (1.1) and $\Delta t=\mathcal{O}(\Delta x^5)$ for Eq. (1.2)). Usually, it is not necessary to choose such a small time step for the purpose of accuracy and is purely an artifact of the explicit time discretization technique. Therefore, implicit methods should be used to improve the computational efficiency.

The discontinuous Galerkin (DG) method is a class of finite element methods using completely discontinuous basis functions, which are usually chosen as piecewise polynomials. Reed and Hill [10] first introduced the DG method in 1973, in the framework of neutron linear transport. For PDEs containing higher order spatial derivatives, the DG method can also be applied directly, Liu and Yan [9] developed direct DG methods for diffusion problems. Then, Bona et al. constructed conservative DG methods for the general KdV equation in [3]. The first LDG method was constructed by Cockburn and Shu in [4] as an extension of the Runge-Kutta DG method to general convection-diffusion problems. The idea of the LDG method is to rewrite the equations with higher order derivatives as a first order system, then apply the DG method to the system. The LDG techniques have been developed for various high order PDEs including the convection diffusion equations [4], nonlinear one-dimensional and two-dimensional KdV type equations [16, 18]. More details about the LDG methods for high-order time dependent PDEs can be found in the review paper [17].

Xia et al. [14] explored the ARK method to solve the stiff ordinary differential equations (ODEs) resulting from an LDG spatial discretization to PDEs with higher order spatial derivatives and found that it was an efficient time discretization method. The implicit method requires to solve system of linear equations at each time step. The efficiency of the method highly depends on the efficiency of the solver for the linear systems. In [14], the resulting linear system of algebraic equations were solved by direct linear solver in LAPACK, which was not efficient for high-dimensional problems. Other traditional iterative methods such as Gauss-Seidel method suffer from slow convergence rates for large scale problems. Thus, we devote to developing an iterative fast solver for the system of linear equations.

The multigrid (MG) method was originally applied to simple boundary value problems, e.g. second-order boundary value problem. Then, the MG method was extended to solve time-dependent PDEs with even-order spatial derivatives and found that it was an efficient method. Recently, the MG method coupled with the DG spatial discretization for the compressible Naiver-Stokes equation [7, 11] and the Euler equation [1, 2] had been studied. In [12, 13], the MG method was introduced to solve the system of algebraic equations arising from the higher order DG discretization of advection dominated flows. Guo