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Near-Field Imaging of Interior Cavities

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Abstract. A novel method is developed for solving the inverse problem of reconstructing the shape of an interior cavity. The boundary of the cavity is assumed to be a small and smooth perturbation of a circle. The incident field is generated by a point source inside the cavity. The scattering data is taken on a circle centered at the source. The method requires only a single incident wave at one frequency. Using a transformed field expansion, the original boundary value problem is reduced to a successive sequence of two-point boundary value problems and is solved in a closed form. By dropping higher order terms in the power series expansion, the inverse problem is linearized and an explicit relation is established between the Fourier coefficients of the cavity surface function and the total field. A nonlinear correction algorithm is devised to improve the accuracy of the reconstruction. Numerical results are presented to show the effectiveness of the method and its ability to obtain subwavelength resolution.

AMS subject classifications: 65N21, 78A46

Key words: Inverse scattering, near-field imaging, Helmholtz equation.

1 Introduction

Inverse scattering problems are concerned with determining the property of the medium by sending an incident wave and measuring the scattered field. These problems arise in a diverse range of applications, including radar, medical imaging, geophysical exploration, and non-destructive testing [12]. In this paper, we consider the problem of reconstructing the shape of an interior cavity by sending and receiving waves from inside. The cavity is represented by a simply connected domain in $\mathbb{R}^2$, and its exterior is assumed to be impenetrable. A time harmonic incident field is generated by a point source placed inside the cavity. Given the incident field, the direct problem is to determine the total field from the known cavity. We are mainly interested in the inverse problem, which is to reconstruct the shape of the cavity from the noisy data of the total field.

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The inverse problem was studied by using either the linear sampling method or the nonlinear integral equation method [23–26]. These work addressed conventional far-field imaging and the resolution was limited by the Rayleigh criterion [13]. Recently novel methods have been developed for solving a class of inverse surface scattering problems in the near-field imaging [3–6, 10, 17]. The surfaces are assumed to be small and smooth perturbations of planar surfaces. The methods require only a single incident field at one frequency, and are shown to be simple, efficient, and stable to reconstruct surfaces with subwavelength resolution. We refer to [7, 15] for near-field imaging of locally perturbed surfaces and [1, 2] for the resolution and stability analysis of a related wave imaging problem.

This work is an extension of near-field imaging from open surfaces to interior cavities. The boundary of the cavity is assumed to be a small and smooth perturbation of a circle centered at the origin. Using the Fourier series expansion, we derive a transparent boundary condition on an artificial circle inside the cavity, and formulate a boundary value problem for the total field. Based on the transformed field expansion, the problem is reduced to a successive sequence of two-point boundary value problems and is solved in a closed form. By dropping higher order terms in the power series expansion, the inverse problem is linearized and a simple reconstruction formula is obtained. Furthermore, a nonlinear correction algorithm is devised to improve the accuracy of the reconstruction. Numerical examples are presented to show the effectiveness of the method. The interior inverse cavity problem has been recently investigated in [27], where a decomposition method was studied to reconstruct the shape of the cavity by using the same data set, i.e., one point source and multiple measurements. The results in this paper are better than those reported in [27]. We refer to [8, 16, 18–22] for the application of transformed field expansion and related boundary perturbation methods for solving some direct scattering problems.

The paper is organized as follows. In Section 2, we introduce the model and formulation of the inverse problem. Section 3 is dedicated to the direct problem where the transformed field expansion is introduced to obtain an analytical solution. The reconstruction formula and the nonlinear correction algorithm are described in Section 4. Numerical examples are presented in Section 5. We conclude the paper in Section 6 with remarks and directions for future research.

2 Model problem

As seen in Fig. 1, the cavity is assumed to be a smooth perturbation of a disk and is defined by the domain

$$\Omega = \{(r,\theta) : 0 \le r < a + f(\theta),\ \theta \in [0,2\pi)\},$$