

A Constrained Finite Element Method Based on Domain Decomposition Satisfying the Discrete Maximum Principle for Diffusion Problems

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Abstract. In this paper, we are concerned with the constrained finite element method based on domain decomposition satisfying the discrete maximum principle for diffusion problems with discontinuous coefficients on distorted meshes. The basic idea of domain decomposition methods is used to deal with the discontinuous coefficients. To get the information on the interface, we generalize the traditional Neumann-Neumann method to the discontinuous diffusion tensors case. Then, the constrained finite element method is used in each subdomain. Comparing with the method of using the constrained finite element method on the global domain, the numerical experiments show that not only the convergence order is improved, but also the nonlinear iteration time is reduced remarkably in our method.

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Key words: Discrete maximum principle, discontinuous coefficients, domain decomposition, constrained finite element method.

1 Introduction

There are many applications of diffusion equations, such as geology components, image dispose and so on. In some multi-material problems, the diffusion tensors are discontinuous, and the computational meshes are usually distorted. So, in practical computations, we have to solve the diffusion equations with discontinuous coefficients on distorted meshes. A good numerical scheme should be not only stable and convergent,

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but also possesses the mathematical property of the physical system, such as positivity-preserving or satisfying the discrete maximum principle (DMP). The maximum principle means, if the source term is non-positive, then the solution attains its maximum on the boundary. A numerical scheme that does not generate spurious extrema in the interior of the domain is said to satisfy the discrete maximum principle, which makes it possible to prove the uniform convergence of an approximate solution to the exact one. In the sufficient conditions of the DMP, M-matrix plays an important role [2, 4]. If the inverse of the coefficient matrix is non-negative, then the solution of the algebraic system is also non-negative for any non-negative right-hand side. A numerical scheme enjoys this property is called positivity-preserving. We should point out that, the DMP criterion is more stringent than the positivity-preserving since it requires that the row sums of the coefficient matrix be zero for all interior nodes.

As we know, it is very difficult to construct a numerical scheme satisfying the discrete maximum principle. Based on the finite volume discretization, the continuous edge flux scheme [10], which needs high regularity, can solve the diffusion problems with smooth coefficients effectively, but fails for the discontinuous coefficients. The multi-point flux approximation (MPFA) methods [11–13] and the mimetic finite difference (MFD) methods [14, 15] are monotone for the shape-regular meshes, but when the diffusion tensors are anisotropic or the meshes are distorted, these methods do not satisfy the positivity-preserving property. The diffusion scheme, which is addressed in [24], is positivity-preserving on various distorted meshes for both smooth and non-smooth solutions. But, from the structure of the algebraic system, we can see that this scheme does not satisfy the discrete maximum principle.

For the finite element methods, the sufficient conditions of the DMP are imposed by severe restrictions on the choice of basis functions and on the geometric properties of the mesh. For example, when the triangulation is acute or non-obtuse, the piecewise-linear finite element solutions of the Poisson equation satisfying the DMP [3]. When the quadrilaterals are non-narrow, the bilinear finite element solutions satisfy the DMP. However, these geometric restrictions fail in the case of high-order finite elements, singularly perturbed convection-diffusion equations and anisotropic diffusion problems. A nonlinear Galerkin finite element method [1] is proposed for isotropic Laplace equation on distorted meshes. Based on repair techniques and constrained optimizations, the methods addressed in [5–7] enforce the linear finite element solution and the mixed element solution satisfying the DMP. Two approaches are addressed in [5]. The first approach is based on the repair technique, which is a posteriori correction of the discrete solution. The second approach is based on the constrained optimization, in which the linear constraints are introduced. In these two approaches, only the global discrete energy of the solution is preserved. We should point out that when the total available energy is less than the total needed one, this repair technique will result in a dead circle and has to terminate, and the solution of the constrained optimization is very expensive as the number of unknowns is increased. In [6], two non-negative mixed finite element formulations for tensorial diffusion equations based on constrained optimization techniques are proposed. The first for-