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C^0IPG for a Fourth Order Eigenvalue Problem

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Abstract. This paper concerns numerical computation of a fourth order eigenvalue problem. We first show the well-posedness of the source problem. An interior penalty discontinuous Galerkin method (C^0IPG) using Lagrange elements is proposed and its convergence is studied. The method is then used to compute the eigenvalues. We show that the method is spectrally correct and prove the optimal convergence. Numerical examples are presented to validate the theory.

AMS subject classifications: 65N25, 65N30, 65N22

Key words: Fourth order eigenvalue problem, discontinuous Galerkin method, spectrum approximation.

1 Introduction

In this paper, we consider a fourth order eigenvalue problem arising in the study of transmission eigenvalues, which have important applications in inverse scattering theory [8, 17, 21]. For the biharmonic equation, there exist quite a few finite element methods in the literature including the conforming elements, partition of unity finite element methods, non-conforming elements, and mixed methods. Construction of high regularity conforming elements is difficult in general [2, 10]. Moreover, they usually involve a large number of degrees of freedom. For example, partition of unity finite element methods [9, 22] are difficult to implement and the resulting linear systems can be severely ill-conditioned. Non-conforming methods such as the Morley element do not have a good

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hierarchy [20] and thus cannot capture smooth solutions efficiently. Mixed methods may produce spurious solutions for non-convex domains [7].

As competitive alternatives, discontinuous Galerkin methods become popular for high order elliptic problems [6, 13, 14]. In particular, in [6, 13], the authors employ an interior penalty Galerkin method using Lagrange elements (C^0IPG) for the biharmonic equation. The method has a good hierarchy and there is no need to enforce the jump of the solution since the finite element space is H^1-conforming. Recently, Brenner et al. employ the C^0IPG to compute biharmonic eigenvalues [7]. Using the classical theory of Babuška and Osborn, they prove the converge of C^0IPG for biharmonic eigenvalue problems and compare it with the Argyris element, the mixed method, and the Morley element.

In this paper, we use the C^0IPG to compute a fourth order problem. Due to the lower order terms, norm convergence of discrete operators is not readily available and thus it is not possible to use the theory of Babuška-Osborn directly as in [7]. To overcome this difficulty, we adopt the method by Antonietti et al. [1] for the Laplace eigenvalue problem, which in turn follows the abstract theory by Descloux, Nassif, and Rappaz [11, 12]. Convergence theory of the finite element methods for eigenvalue problems has been studied by many researchers since 1970s. We refer the readers to the book chapter by Babuška and Osborn [3] and the references therein for studies before 1991. For recent developments, we refer the readers to the survey paper [4].

The rest of the paper is arranged as follows. In Section 2, we present a fourth order eigenvalue problem with low order terms and prove the well-posedness. Section 3 describes the C^0IPG for the source problem and shows some useful properties of the discrete problem. We develop the convergence theory for the eigenvalue problem in Section 4. Numerical examples are given in Section 5.

2 A fourth order eigenvalue problem

Let Ω be a bounded Lipschitz polygonal domain in $\mathbb{R}^2$ with unit outward normal n. Let $m(x)$ be a bounded smooth function such that $m(x)>\gamma>0$ and γ,τ be positive constants. In addition, let $\|\cdot\|$ denote the L^2 norm and C,C_1,C_2 denote generic constants.

We consider a fourth order eigenvalue problem of finding μ and u such that

$$(\triangle+\tau)m(x)(\triangle+\tau)u+\tau^2u=\mu\triangle u \quad \text{in } \Omega, \tag{2.1}$$

with boundary conditions

$$u=0, \quad \frac{\partial u}{\partial n}=0 \quad \text{on } \partial\Omega. \tag{2.2}$$

The corresponding source problem can be stated as follows: given f, find u such that

$$(\triangle+\tau)m(x)(\triangle+\tau)u+\tau^2u=\triangle f, \tag{2.3}$$

with the boundary conditions (2.2).