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Effective Boundary Conditions: A General Strategy and Application to Compressible Flows Over Rough Boundaries

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Abstract. Determining the drag of a flow over a rough surface is a guiding example for the need to take geometric micro-scale effects into account when computing a macro-scale quantity. A well-known strategy to avoid a prohibitively expensive numerical resolution of micro-scale structures is to capture the micro-scale effects through some *effective boundary conditions* posed for a problem on a (virtually) smooth domain. The central objective of this paper is to develop a numerical scheme for accurately capturing the micro-scale effects at essentially the cost of twice solving a problem on a (piecewise) *smooth* domain at affordable resolution. Here and throughout the paper "smooth" means the absence of any micro-scale roughness. Our derivation is based on a "conceptual recipe" formulated first in a simplified setting of boundary value problems under the assumption of sufficient local regularity to permit asymptotic expansions in terms of the micro-scale parameter.

The proposed multiscale model relies then on an upscaling strategy similar in spirit to previous works by Achdou et al. [1], Jäger and Mikelic [29, 31], Friedmann et al. [24, 25], for *incompressible* fluids. Extensions to *compressible* fluids, although with several noteworthy distinctions regarding e.g. the "micro-scale size" relative to boundary layer thickness or the systematic treatment of different boundary conditions, are discussed in Deolmi et al. [16,17]. For proof of concept the general strategy is applied to the compressible Navier-Stokes equations to investigate steady, laminar, subsonic flow over a flat plate with partially embedded isotropic and anisotropic periodic roughness imposing adiabatic and isothermal wall conditions, respectively. The results are compared with high resolution direct simulations on a fully resolved rough domain.

AMS subject classifications: 74Q15, 76G25, 35Q30

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1 Introduction

From several scenarios in nature it is well-known that microstructures on surfaces can significantly reduce drag. For instance, the skin of a shark exhibits small-scale structures that makes the shark a very fast maritime hunter [59]. This has been confirmed by experiments conducted in oil channels to study biological surfaces, e.g., shark-skin replicas, hairy surfaces such as seal fur, [53–55], experiencing significant drag reduction. Such observations lead engineers to mimic this effect for economical and ecological reasons in practical applications such as aviation. For instance, in [56, 57] riblets are shown to reduce the overall drag of airfoils and aircraft provided the riblet spacing is chosen appropriately. For flight tests of an Airbus 320 drag reduction was observed, see [58], but not as significant as for experiments in wind tunnels and oil channels, respectively. For a review on drag reduction using riblets we refer to [50–52].

1.1 Objectives

To gain a deeper insight in the underlying physical mechanisms of drag reduction and eventually permit predictions, simulations are performed that complement experimental investigations. Since resolving the microstructures requires a high resolution, numerical simulations are very expensive and, depending on the flow regime, are only feasible for small configurations. For a real application such as an airfoil the computational cost will be prohibitively high and a simulation will not be feasible in spite of an ever increasing computer power.

To deal with this type of problems a natural strategy is to resort to model reduction concepts. Some well-known strategies are *homogenization techniques* [10, 33, 48], *(heterogeneous) multiscale modeling* [21, 22] and multiscale finite element methods [20], all aiming to quantify the influence of small scale effects on the resolved macroscopic scale *without* directly resolving small scale structures. Typically, these concepts need to be adapted to the problem at hand.

In fact, it should be noted that, strictly speaking, for the problems under consideration there is no clear (physical) scale separation so that a straightforward application of the heterogeneous multi-scale method is delicate. Rather the range of relevant scales is too large to be resolved.

The central objective in the present paper is to develop a new *computational* model reduction strategy that differs from the aforementioned methods. Our starting point is the formulation of an upscaling strategy where the micro-scale effect of a structured rough surface is modeled by means of effective boundary conditions given on a virtually smooth wall. For the derivation of these conditions the exact solution of the original problem on the rough domain is expanded in a *zeroth order solution* depending only on the macro-scale, i.e., the flow equations are solved in the artificial smooth domain, and an upscaling term that depends on macro-scale *and* micro-scale variables in order to capture the micro-scale effects suppressed in the zeroth order solution. A natural idea is to plug this