Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2016-0080

Vol. **22**, No. 2, pp. 375-392
August 2017

A Symmetric Direct Discontinuous Galerkin Method for the Compressible Navier-Stokes Equations

Huiqiang Yue, Jian Cheng and Tiegang Liu*

School of Mathematics and Systems Science, Beihang University, Beijing 100191, China.

Communicated by Chi-Wang Shu

Received 30 May 2016; Accepted (in revised version) 10 January 2017

Abstract. In this work, we investigate the numerical approximation of the compressible Navier-Stokes equations under the framework of discontinuous Galerkin methods. For discretization of the viscous and heat fluxes, we extend and apply the symmetric direct discontinuous Galerkin (SDDG) method which is originally introduced for scalar diffusion problems. The original compressible Navier-Stokes equations are rewritten into an equivalent form via homogeneity tensors. Then, the numerical diffusive fluxes are constructed from the weak formulation of primal equations directly without converting the second-order equations to a first-order system. Additional numerical flux functions involving the jump of second order derivative of test functions are added to the original direct discontinuous Galerkin (DDG) discretization. A number of numerical tests are carried out to assess the practical performance of the SDDG method for the two dimensional compressible Navier-Stokes equations. These numerical results obtained demonstrate that the SDDG method can achieve the optimal order of accuracy. Especially, compared with the well-established symmetric interior penalty (SIP) method [18], the SDDG method can maintain the expected optimal order of convergence with a smaller penalty coefficient.

AMS subject classifications: 35Q30, 65M60, 76N15

Key words: Compressible Navier-Stokes equations, discontinuous Galerkin method, compressible viscous flow.

1 Introduction

In recent years, discontinuous Galerkin (DG) method has been widely used in various problems since it was first introduced by Reed and Hill [26] and then vigorously developed by Cockburn, Shu and co-workers in a series of papers [10–12, 14]. Compared with

*Corresponding author. *Email addresses:* yuehq@buaa.edu.cn (H. Yue), chengjian@buaa.edu.cn (J. Cheng), liutg@buaa.edu.cn (T. Liu)

classical finite element (FE) method and finite volume (FV) method, DG method possesses some attractive properties [8], such as flexibility in handling hp-adaptive strategies, complex geometries and parallelization.

However, the application of DG method to diffusion problems is far less certain and more challenging [27] due to the fact that the derivatives of solution have to be evaluated at interfaces. The first attempt to apply DG method to discretize second-order partial differential operators dated back to the interior penalty (IP) method which was independently proposed and studied in late 1970s [1,15]. For the compressible Navier-Stokes (NS) equations, the first method was proposed by Bassi and Rebay (BR1) in 1997 [3], later, in order to maintain the compactness and stability of DG method an improved method called BR2 scheme was developed by Bassi and Rebay based on BR1 formula [2]. Hartmann and Houston proposed the symmetric interior penalty (SIP) [18] method which can guarantee the optimal convergence order in L_2 error analysis not only for the whole solution but for the output functionals. The compact DG (CDG) method introduced by Peraire and Persson [25] as a variation of local DG (LDG) [13] was also applied to the compressible Navier-Stokes equations. Based on the smooth reconstructed solution Luo et al. [23] proposed a reconstructed DG (rDG) method on arbitrary grids for compressible flows. In order to reduce the computational costs of DG methods, a kind of hybridizable DG (HDG) method was developed by Cockburn and his colleagues in [9]. In HDG method, additional unknowns defined on interior faces were introduced, then, a linear system only involving degrees of freedoms (DOFs) on interior faces was obtained. In general, the size of this linear system was smaller than which obtained via DG method. Later, Peraire and Nguyuen extended this method to compressible flows [24].

Recently, a direct DG (DDG) method was originally proposed by Liu and Yan [21,22] for scalar diffusion problems based on the direct weak formulation of primal equations without converting the second-order equations to a first-order system. The numerical flux defined by the DDG method is simple, compact, conservative, and consistent. The most remarkable feature of the DDG method is its simplicity in implementation and its efficiency in computational cost. In their paper, the DDG method was successfully applied to one and two dimensional diffusion and convection-diffusion equations on structured meshes. Later, Jian Cheng and his co-workers successfully extended and applied the DDG methods to solve the compressible Navier-Stokes equations [5,7] and Reynolds-averaged Navier-Stokes equations [6] for both laminar and turbulent flows on arbitrary grids. Based on their numerical results, DDG method can achieve the designed order of accuracy and is able to deliver the same accuracy as the widely used BR2 method at a significantly reduced cost, clearly demonstrating that the DDG method provides an attractive alternative for solving the compressible Navier-Stokes equations on arbitrary grids owning to its simplicity in implementation and its efficiency in computational cost.

Very Recently, a symmetric direct DG (SDDG) method is further introduced and developed by Vidden and Yan [19, 29] for diffusion equations. Compared to the original DDG method, a numerical flux for the test function derivative is introduced, thus, includes more interface terms in SDDG method. The symmetric structure is the key in