Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2016-0201

Vol. **22**, No. 2, pp. 494-516
August 2017

Adaptive Stokes Preconditioning for Steady Incompressible Flows

Cédric Beaume*

School of Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom.

Received 16 November 2016; Accepted (in revised version) 19 December 2016

Abstract. This paper describes an adaptive preconditioner for numerical continuation of incompressible Navier–Stokes flows based on Stokes preconditioning [42] which has been used successfully in studies of pattern formation in convection. The preconditioner takes the form of the Helmholtz operator $I-\Delta tL$ which maps the identity (no preconditioner) for $\Delta t \ll 1$ to Laplacian preconditioning for $\Delta t \gg 1$. It is built on a first order Euler time-discretization scheme and is part of the family of matrix-free methods. The preconditioner is tested on two fluid configurations: three-dimensional doubly diffusive convection and a two-dimensional projection of a shear flow. In the former case, it is found that Stokes preconditioning is more efficient for $\Delta t = \mathcal{O}(1)$, away from the values used in the literature. In the latter case, the simple use of the preconditioner is not sufficient and it is necessary to split the system of equations into two subsystems which are solved simultaneously using two different preconditioners, one of which is parameter dependent. Due to the nature of these applications and the flexibility of the approach described, this preconditioner is expected to help in a wide range of applications.

AMS subject classifications: 35Q30, 35Q35, 37M20, 37N10, 65F08, 65P30, 76D05, 76E05, 76E06

Key words: Fluid dynamics, hydrodynamic stability, numerical continuation, preconditioning, Stokes preconditioning, doubly diffusive convection, shear flows.

1 Introduction

The development of specialized numerical methods and the increase in available computing resources have helped to make substantial progress in understanding many nonlinear problems as dynamical systems. The most basic tool available to that end is time integration which simulates the temporal evolution of an initial condition, thereby emulating an experimental or natural realization. Time integration provides access to the preferred transient and end state, however, it does not (necessarily) provide access to information regarding the origin of these end states. One way to understand how these

*Corresponding author. *Email address:* `c.m.l.beaume@leeds.ac.uk` (C. Beaume)

states are formed is to compute unstable solutions. Unstable states cannot be obtained (or in some rare cases with great difficulty) using time integration but help provide a complete picture of the dynamical system: they can gain stability or lead to the creation of new solutions or new transients under parametric changes. Numerical continuation has been developed to complement time-integration in that respect and has become an essential part of the toolkit of the nonlinear dynamicist.

Pioneered by Keller [26], these methods help compute steady solutions of a system of ordinary differential equations (ODE) and their evolution as parameter values are changed. They are designed to *continue* a fixed point in parameter space in order to draw its *branch* and unfold the *bifurcation diagram*, thereby revealing its formation mechanisms. Continuation methods typically consist in a two-step algorithm comprising a prediction phase based on previous iterates along the branch and a correction phase involving a fixed point method [39]. Due to their nature, these methods can compute exact solutions regardless of their stability and provide information on the effect of parametric changes on them. Numerical continuation has become a popular tool, broadly used in many different fields [27, 40] and a myriad of packages have been developed and released in the public domain [14, 17, 18, 28, 44].

Fluid dynamics has seen much progress with the help of continuation methods. Intricate pattern formation problems have been elucidated such as that of Rayleigh–Bénard convection rolls in Cartesian [41], cylindrical [11] and spherical shell geometries [20], doubly diffusive convection [10] and free surface binary fluid convection [9]. Spatially localized pattern formation, involving large aspect-ratio domains, has also been investigated with great success: a collection of spatially localized convective states has been found in two-dimensional large aspect-ratio binary fluid convection [33,34], rotating convection [3, 7] and magnetoconvection [29]. Despite the successful and reliable use of continuation methods in two-dimensional and small three-dimensional domains, the extension to more complex geometries constitutes a major challenge. The most noticeable attempts concern doubly diffusive convection in a three-dimensional domain of square cross section and large transverse direction [5, 8] and porous medium convection in domains extended in two directions [30], each of these problems involving $\mathcal{O}(10^6)$ degrees of freedom. These studies involved long simulations and require a high level of experience in the use of numerical continuation.

Another area of fluid dynamics that has benefited from the developments of numerical continuation is that of transition to turbulence. Shear flows such as plane Couette flow or pipe flow are subcritical flows, i.e., the trivial laminar solution is stable and coexists with turbulence, a state in which the flow displays spatial and temporal complexity, above a threshold value of the parameter. The pioneering discovery of unstable nonlinear solutions in plane Couette flow [35] has drawn a great deal of attention and meticulous studies of this unstable state have provided crucial understanding of transition in small domains [25]. Numerical continuation has also led to the discovery of a number of new solutions [24, 32, 46] that taken together provide a comprehensive picture of transitional phenomena. Similar studies have taken place in other shear flows and hinted at a com-