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REVIEW ARTICLE

Heterogeneous Multiscale Methods: A Review

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Abstract. This paper gives a systematic introduction to HMM, the heterogeneous multiscale methods, including the fundamental design principles behind the HMM philosophy and the main obstacles that have to be overcome when using HMM for a particular problem. This is illustrated by examples from several application areas, including complex fluids, micro-fluidics, solids, interface problems, stochastic problems, and statistically self-similar problems. Emphasis is given to the technical tools, such as the various constrained molecular dynamics, that have been developed, in order to apply HMM to these problems. Examples of mathematical results on the error analysis of HMM are presented. The review ends with a discussion on some of the problems that have to be solved in order to make HMM a more powerful tool.

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Contents

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1 Introduction

The heterogeneous multiscale method, or HMM, proposed in [56] is a general framework for designing multiscale methods for a wide variety of applications. The name "heterogeneous" was used to emphasize that the models at different scales may be of very different nature, e.g. molecular dynamics at the micro scale and continuum mechanics at the macro scale. Since its inception, there has been substantial progress on multiscale modeling using the philosophy of HMM. The HMM framework has proven to be very useful in guiding the design and analysis of multiscale methods, and in several applications, it helps to transform multiscale modeling from a somewhat ad hoc practice to a systematic technique with a solid foundation. Yet more possibilities are waiting to be explored, particularly in the application areas. Many new questions of physical, numerical or analytical nature have emerged. All these make HMM an extremely fruitful and promising area of research.

The purpose of this article is to give a coherent summary of the status of HMM. It is our hope that this summary will help the reader to understand the design principle behind the HMM philosophy, the main obstacles that one has to overcome when using HMM for a particular problem, and the immediate problems that have to be solved in order to make HMM a more powerful tool.

Just what *is* HMM? After all many multiscale modeling strategies discussed in the applied communities are heterogeneous in nature, i.e. they involve models of different nature at different scales, so what is special about HMM? Through this review we will show that HMM is a general framework for designing multiscale methods that can be applied to a wide variety of applications. In a nutshell, the philosophy is as follows. Assume we are interested in studying the macroscale behavior of a problem for which the macroscale model is only partly known or is valid only on part of the physical domain. In typical situations, we either lack the detailed constitutive relation, or the macroscale model is invalid due to the presence of defects or localized singularities. Assume, on the other hand, that we do have an accurate microscale model at our disposal, but it is too expensive to abandon the macroscale model completely and only use the microscale