Quinpi: Integrating Stiff Hyperbolic Systems with Implicit High Order Finite Volume Schemes

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Abstract. Many interesting physical problems described by systems of hyperbolic conservation laws are stiff, and thus impose a very small time-step because of the restrictive CFL stability condition. In this case, one can exploit the superior stability properties of implicit time integration which allows to choose the time-step only from accuracy requirements, and thus avoid the use of small time-steps. We discuss an efficient framework to devise high order implicit schemes for stiff hyperbolic systems without tailoring it to a specific problem. The nonlinearity of high order schemes, due to space- and time-limiting procedures which control nonphysical oscillations, makes the implicit time integration difficult, e.g. because the discrete system is nonlinear also on linear problems. This nonlinearity of the scheme is circumvented as proposed in (Puppo et al., Comm. Appl. Math. & Comput., 2023) for scalar conservation laws, where a first order implicit predictor is computed to freeze the nonlinear coefficients of the essentially non-oscillatory space reconstruction, and also to assist limiting in time. In addition, we propose a novel conservative flux-centered a-posteriori time-limiting procedure using numerical entropy indicators to detect troubled cells. The numerical tests involve classical and artificially devised stiff problems using the Euler's system of gas-dynamics.

AMS subject classifications: 65M08, 65M20, 35L65, 65L04 **Key words**: Implicit methods, essentially non-oscillatory schemes, finite volumes, hyperbolic systems, entropy indicators.

1 Introduction

Mathematical models for the description of fluids, plasmas, and many other physical phenomena, are typically given in terms of systems of hyperbolic conservation laws.

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These systems are characterized by a set of multi-dimensional partial differential equations (PDEs) that express the conservation of various physical quantities in terms of their respective fluxes. A prototypical example is provided by the Euler's equations for gasdynamics describing the conservation of mass, momentum, and energy of a gas.

In this work we focus on one-dimensional systems of $m \ge 1$ hyperbolic conservation laws:

$$\frac{\partial}{\partial t}\mathbf{u}(x,t) + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}(x,t)) = \mathbf{0}, \tag{1.1}$$

where, $\mathbf{u} : \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}^m$ is the quantity of interest, and $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^m$ is the vector of the flux functions. System (1.1) is hyperbolic when the eigenvalues $\{\lambda_j(\mathbf{u}(x,t))\}_{j=1}^m$ of the associated Jacobian matrix are real and determine a complete set of eigenvectors. The eigenvalues of (1.1) provide the characteristic velocities, which describe the propagation speed of waves in the system. These waves can be either acoustic waves (shocks and rarefactions) or material waves (contact discontinuities). Requiring that the eigenvalues are real implies that the propagation speed of information through the system is finite.

Solving hyperbolic systems of conservation laws is a challenging task, both analytically and numerically, e.g. due to the occurrence of singularities or the need of devising high order accurate non-oscillatory methods to avoid low-resolution approximations. Another source of numerical difficulty is represented by stiff problems that occur when the system is characterized by speeds spanning different orders of magnitude, namely when $\frac{\max_{j=1,\dots,m}|\lambda_j(\mathbf{u})|}{\min_{j=1,\dots,m}|\lambda_j(\mathbf{u})|} \gg 1$. This happens, for instance, in gas-dynamics when the fluid speed is much less than the speed of the acoustic waves. In many applications the phenomenon of interest travels with a low speed. An example is provided by low-Mach number problems occurring when the equations governing the flow become stiff due to the very low fluid velocity compared to the speed of sound in the fluid. In these situations, the compressibility effects of the fluid can be neglected, and the fluid is almost incompressible. Then, if the interest is on the movement of the fluid, accuracy in the propagation of sound waves becomes irrelevant. For low-Mach problems we refer to [1, 11, 21-23, 49].

Numerical schemes used to solve hyperbolic problems need to be carefully designed to handle the stiff regime. In fact, it is well-known that explicit schemes are subject to the Courant-Friedrichs-Levy (CFL) stability condition that specifies a constraint on the numerical speed in relation to the maximum speed of information propagating in the system. More precisely, let Δt and h be the time-step and the mesh width of a numerical scheme, respectively. We define the *numerical speed* as $s_n = h/\Delta t$ which approximates the speed with which the numerical data propagate in the discretized system. Then, the CFL condition imposes that s_n must be faster than the maximum speed of propagation to ensure that information does not travel too far between adjacent space cells during one time-step. For this reason, the stability request on the time-step of explicit schemes becomes very restrictive for stiff problems due to the presence of fast waves, thus limiting the computational efficiency of the scheme. In contrast, implicit schemes can have supe-

rior stability properties. Therefore, they can be less constrained by the CFL condition and can be employed with a time-step focusing on the phenomenon of interest, for instance, on the fluid speed, thus potentially allowing for larger time-step sizes. In addition, since the accuracy of a scheme depends on the difference between the numerical and the actual speed, it turns out that implicit schemes reduce accuracy on the (fast) acoustic waves, still resulting highly accurate on the (slow) material waves. However, implicit methods are computationally more expensive than explicit ones since they require the solution of a system of equations, in general nonlinear, at each time-step. Therefore, the choice between explicit and implicit schemes depends on the specific problem being solved and the desired trade-off between computational efficiency and accuracy.

Here, we deal with an efficient formulation of implicit high order finite volume schemes [37]. In first order implicit schemes the only source of nonlinearity is due to the nonlinear flux function, namely to the physical structure of the model, which is therefore unavoidable; however they produce large dissipation errors. High order accurate implicit schemes require, as much as their explicit counterparts, nonlinear space-limiting procedures to prevent spurious oscillations (see e.g. [43] for a discussion of the TVD property of implicit schemes). Such space-limiting procedures introduce an additional source of nonlinearity which becomes computationally challenging when using implicit schemes. The novel idea of [43] was to simplify considerably the implicit high order scheme by using a first order predictor to freeze the non-linearities of the space-limiting procedure. The implicit approach proposed in [43], named Quinpi, was tailored to the third order implicit approximation of scalar conservation laws. More specifically, third order accuracy was achieved by using a third order Diagonally Implicit Runge-Kutta (DIRK) for the time integration and a third order Central Weighted Essentially Non-Oscillatory (CWENO) reconstruction, cf. [19], for the space discretization. The first order implicit scheme was based on a composite backward Euler, evaluated at the abscissae of the DIRK, naturally combined with a piecewise constant (i.e. linear in the data) reconstruction in space. This predictor was used to freeze the nonlinear weights of the CWENO reconstruction making the resulting third order implicit scheme nonlinear just because of the nonlinearity of the flux function.

As noted in [4, 43], the appearance of spurious oscillations can be observed in implicit integration, especially for large Courant numbers, despite space-limiting being performed. A *time-limiting* procedure is required in order to control the oscillations arising in the implicit time integration, i.e. when the large time-step size allows for propagation of waves crossing several adjacent space cells during a single time-step. In this context, "time-limiting" refers to the practice of downgrading the order of the solution on irregular cells, thereby mitigating spurious oscillations associated with the use of large time-steps. The problem of the time-limiting has been discussed in several papers, which typically deal with limiting in space and time simultaneously. For instance, we mention [24, 25] for second-order schemes and [4] for a fully nonlinear third order implicit scheme. In [43], instead, the time-limiting was obtained as a cell-centered a-posteriori nonlinear blending of the first order and third order solutions. Troubled cells were detected using a combination of space and time regularity indicators, in a WENO-like fashion. However, being cell-centered, the technique in [43] has the drawback of being nonconservative, and, thus, it requires a conservative correction. Conservative flux-centered time-limiting procedures, inspired by the Multi-dimensional Optimal Order Detection (MOOD) method introduced in [13,14], were proposed in [26,50] for the control of nonphysical oscillations of high order implicit numerical solutions. MOOD was originally developed in order to reduce the order of the space reconstruction on problematic cells with the help of several problem-dependent detectors to check whether extrema of the numerical solution are smooth, physical or spurious oscillations. MOOD was also extended to other contexts, as for instance in [39,45,51].

The contribution of the present work is twofold:

- We extend the Quinpi framework proposed in [43] to the case of the numerical approximation of general stiff hyperbolic systems when the interest is on the movement of the fluid. This implicit approach is not tailored to the solution of a specific stiff problem, as it usually happens when devising numerical schemes for low-Mach problems. Moreover, the space-limiting exploits the novel CWENO reconstruction [46] which does not make use of ghost cells for boundary reconstructions;
- 2. We introduce a flux-centered conservative time-limiting procedure inspired by the MOOD technique, namely we replace the high order numerical fluxes at the cell interfaces of troubled cells with low order numerical fluxes. A crucial point is the choice of the troubled cells indicator; here we investigate the use of the numerical entropy production error [40,41] as indicator that signals non-smooth solutions instead of the typical MOOD detectors of oscillatory cells.

Finally, we mention the following approaches to the implicit integration of hyperbolic systems developed in the literature. A fifth order implicit WENO scheme was proposed in [32], where a predictor-corrector technique was also used. However, the predictor was based on an explicit first order scheme and therefore it allows to deal with small Courant numbers only. A fully nonlinear implicit scheme, based on a third order RADAU time integrator and a third order WENO reconstruction, can be found in [4]. Fully implicit, semi-implicit, implicit-explicit, local time-stepping and active flux treatments of stiff hyperbolic equations were also investigated, e.g., in [5,7,10,15–18,27,28].

The paper is organized as follows. In Section 2 we introduce the fully implicit third order scheme of system (1.1) which uses the CWENO reconstruction of [46] for the space approximation and a DIRK method for the time integration. Then, in Section 3, we devise the Quinpi framework, which allows to overcome the nonlinearity of CWENO by freezing the nonlinear weights with a first order composite backward Euler approximation of the solution. The limiting in time is discussed, which employs a flux-centered MOOD technique combined with the numerical entropy production error as detector of troubling cells. Numerical experiments are provided in Section 4. Precisely, Section 4.1 and Section 4.2 are focused on the Euler's equations for gas-dynamics, where we investigate the experimental order of convergence of the Quinpi scheme and its performance

on stiff and low-Mach problems. Section 4.3 is devoted to the comparison of the novel time-limiting procedure presented in this work, the cell-centered one of [43] and the flux-centered of [50]. Finally, we discuss results and perspectives in Section 5.

2 Third order space-time fully implicit discretization

We consider a finite volume approximation of (1.1) through the method of lines (MOL) on the compact computational domain $\Omega = [a,b] \subset \mathbb{R}$. To this end, we discretize Ω with N uniform cells $\Omega_j = [x_j - h/2, x_j + h/2]$ of amplitude h > 0, such that $\bigcup_{j=1}^N \Omega_j = \Omega$ and $x_j = a + (j-1/2)h$ are the cell centers. For the sake of simplicity, we will describe the scheme for a uniform grid, but it is easy to generalize the scheme to a non uniform mesh. Defining the cell averages of the exact solution on a given space cell Ω_j as

$$\overline{\mathbf{u}}_{j}(t) = \frac{1}{h} \int_{\Omega_{j}} \mathbf{u}(x,t) \mathrm{d}x, \quad t \ge 0$$

the MOL provides the following semi-discrete form of system (1.1):

$$\frac{\mathrm{d}\overline{\mathbf{u}}_{j}(t)}{\mathrm{d}t} = -\frac{1}{h} \left[\mathbf{f} \left(\mathbf{u}_{j+\frac{1}{2}}(t) \right) - \mathbf{f} \left(\mathbf{u}_{j-\frac{1}{2}}(t) \right) \right], \quad j = 1, \cdots, N, \quad t \ge 0, \tag{2.1}$$

where $\mathbf{u}_{j\pm 1/2}(t) = \mathbf{u}(x_j \pm h/2, t)$. System (2.1) describes the conservation of the cell averages as the difference of the right and left fluxes at the cell boundaries. Up to now, no numerical approximation of the exact solution has been introduced. In fact, (2.1) is still exact.

In order to transform the MOL in a numerical scheme, first one has to transform the exact system (2.1) in a closed system for the cell averages and, afterwards, one can introduce a time discretization of the resulting coupled ODE system which evolves in time the cell averages. Consequently, there is the problem of the knowledge of point values of the solution at the cell interfaces for the evaluation of the flux function. The extrapolation of these values from the cell averages is the so-called *reconstruction problem*. WENO and CWENO schemes, see e.g. [19,33,38,48], and their developments [2,6,12,47], are examples of numerical procedures computing point values as function of the cell averages. The advantage of WENO and CWENO schemes is that they achieve high order approximations of the reconstructions, but they pay the price of enlarging the stencil, compared to low order schemes, and of being highly nonlinear, which is a computational bottleneck in implicit time integration.

In the following, before dealing with the time integration of (2.1), we recall the space reconstruction based on a recently developed CWENO scheme [46].

2.1 Space reconstruction: Third order CWENOZ without ghost cells

In reconstruction procedures, the goal is to provide a space limited approximation of the exact solution $\mathbf{u}(\cdot, t)$, at a given time $t \ge 0$, using the knowledge of its cell averages. Since

A CWENO type reconstruction defines an approximation of $u(\cdot, t)$ as

$$u(x,t) \approx \sum_{j=1}^{N} R_j(x;t) \chi_{\Omega_j}(x), \quad t \ge 0,$$

where χ_{Ω_j} is the characteristic function of the cell Ω_j , and $R_j(x;t)$ is the space reconstruction polynomial for $x \in \Omega_j$, which depends on time through the time-dependent cell averages. Indeed, here the time variable t is treated as a parameter meaning that the space reconstruction polynomial is computed at a fixed time. If the desired reconstruction point, say \hat{x} , lies within the cell Ω_j , then the evaluation of the polynomial $R_j(\hat{x};t)$ provides the needed point values of $u(\hat{x},t)$. The CWENO type procedure differs from the classical WENO scheme by the fact that each polynomial $R_j(x;t)$ is globally defined in its reference cell Ω_j and, therefore, it can be pre-computed and later evaluated at the needed locations.

We focus on third order space reconstructions. Then, the CWENO scheme with Z-type nonlinear weights [12, 20] defines the polynomial $R_j(x;t)$ for $x \in \Omega_j$ as follows.

Definition 2.1 (Third order CWENOZ reconstruction, see [20]). Let $P_{opt} \in \mathbb{P}^2$ be the *optimal* polynomial of degree 2, which interpolates all the data in the three-cell stencil $S_{opt} = {\Omega_{j-1}, \Omega_j, \Omega_{j+1}}$. Further, let $P_L, P_R \in \mathbb{P}^1$ be polynomials of degree 1 such that P_L interpolates the cell averages of the left-biased sub-stencil $S_L = {\Omega_{j-1}, \Omega_j}$, and P_R interpolates the cell averages of the right-biased sub-stencil $S_R = {\Omega_j, \Omega_{j+1}}$. Let also ${d_0, d_L, d_R}$ be a set of strictly positive real coefficients such that $\sum_{k=0,L,R} d_k = 1$.

The CWENOZ procedure computes the reconstruction polynomial on Ω_i as

$$R_j^{\mathsf{CWZ}}(x;t) = \frac{\omega_0}{d_0} \left(P_{\mathsf{opt}}(x;t) - \sum_{k=L,R} d_k P_k(x;t) \right) + \sum_{k=L,R} \omega_k P_k(x;t) \in \mathbb{P}^2,$$
(2.2)

where ω_0 , ω_L and ω_R are the (nonlinear) coefficients defined as

$$\alpha_k = d_k \left(1 + \left(\frac{\tau}{I_k + \epsilon} \right)^p \right), \quad \omega_k = \frac{\alpha_k}{\sum_{i=L,0,R} \alpha_i}, \quad k = 0, L, R.$$
(2.3)

In (2.3), I_0 , I_L and I_R are the regularity indicators of the associated polynomials P_L , P_{opt} and P_R , respectively, computed as the Jiang-Shu indicators from [33]:

$$I[P_k] := \sum_{i=1}^{\deg(P_k)} h^{2i-1} \int_{\Omega_j} \left(\frac{\mathrm{d}^i}{\mathrm{d}x^i} P_k(x;t) \right)^2 \mathrm{d}x, \quad k = 0, L, R.$$
(2.4)

Finally, $\epsilon = h^q$, $q \ge 1$, $p \ge 1$ and τ is the following global smoothness indicator

$$\tau := |2I_0 - I_L - I_R|. \tag{2.5}$$

The CWENOZ reconstruction polynomial switches between the high accurate polynomial P_{opt} , when the cell averages in the stencil S_{opt} are a sampling of a smooth enough function, and a nonlinear blending of P_{opt} and of the lower degree polynomials P_L, P_R when a discontinuity is present in the stencil S_{opt} .

The use of the Z-type weights (2.3) allows to have better accuracy on smooth data compared to classical weights [19, 38, 48], especially on coarse grids, without sacrificing the non-oscillatory properties. This is obtained by using the optimal choice (2.5) of the global smoothness indicator that makes τ much smaller than the regularity indicators when the data in S_{opt} are smooth enough. In [20], it is proven that the accuracy of the CWENOZ reconstruction on smooth flows is the optimal one, provided that deg(P_{opt}) $\leq 2 \text{deg}(P_k)$, for k = L, R.

Close to boundaries, the central stencil S_{opt} may not be defined, because it would not be fully contained in Ω . In this case, one can consider for P_{opt} a 3-cell stencil entirely biased towards the domain interior, but the need of controlling spurious oscillations requires the inclusion of a polynomial $\tilde{P} \in \mathbb{P}^0$, defined on the cells which contain the endpoints of the physical domain. Indeed, assume that the 3-cell stencil is given by $\{\Omega_1, \Omega_2, \Omega_3\}$, then \tilde{P} allows to select the smooth part of the stencil when a discontinuity is present either in Ω_2 or Ω_3 . Optimal accuracy can still be achieved provided that the corresponding linear weight is infinitesimal of order $\mathcal{O}(h^r)$, for some r > 0. This approach was introduced for CWENOZ type reconstructions in [47], where a thorough study of sufficient conditions on r, and on the other parameters of the scheme, to achieve optimal accuracy has been performed, and exploited for reconstructions free of ghost cells in [46]. In the following definition, we recall a particular third order CWENOZ Adaptive Order reconstruction which will be used to define the boundary reconstruction. Again, we consider the one-dimensional case. For two-dimensional reconstructions we refer to [46].

Definition 2.2 (Third order CWENOZ reconstruction without ghost cells, see [46]). Let $R_i(x;t)$ be the reconstruction polynomial related to the cell Ω_i . Then,

$$R_{j}(x;t) = \begin{cases} R_{j}^{\mathsf{CWZ}}(x;t), & j = 2, \cdots, N-1, \\ R_{j}^{\mathsf{AO}}(x;t), & j = 1 \text{ with } \mathcal{S}_{\mathsf{opt}} = \{\Omega_{1}, \Omega_{2}, \Omega_{3}\}, \\ R_{j}^{\mathsf{AO}}(x;t), & j = N \text{ with } \mathcal{S}_{\mathsf{opt}} = \{\Omega_{N-2}, \Omega_{N-1}, \Omega_{N}\}, \end{cases}$$
(2.6)

where $\tau_j = \tau$ in (2.5), for $j = 2, \dots, N-1$, and $\tau_1 = \tau_2, \tau_N = \tau_{N-1}$.

The reconstruction in the first and last computational cells appearing in (2.6) are defined exploting the results in [47], which we report here for completeness in the third order case.

Definition 2.3 (Third order CWENO-AO reconstruction, see [47]). Let $P_{opt} \in \mathbb{P}^2$ be the *optimal* polynomial of degree 2 which interpolates all the given data in the three-cell stencil S_{opt} such that $\Omega_j \in S_{opt}$. Let $P \in \mathbb{P}^1$ be the polynomial of degree 1 such that P interpolates

the cell averages of a two-cell sub-stencil S such that $\Omega_j \in S \subset S_{opt}$. Further, let $\tilde{P} \in \mathbb{P}^0$ be the constant polynomial associated to the sub-stencil $\tilde{S} = {\Omega_j}$, namely $\tilde{P}(x;t) = \bar{u}_j$ for $x \in \Omega_j$. Let also ${d_0, d, \tilde{d}}$ be a set of strictly positive real coefficients such that $d_0 + d + \tilde{d} = 1$ and $\tilde{d} = h^r$ for some r > 0.

The CWENOZ-AO reconstruction polynomial on Ω_i is

$$R_j^{AO}(x;t) = \frac{\omega_0}{d_0} \left(P_{opt}(x;t) - dP(x;t) - \tilde{d}\tilde{P}(t) \right) + \omega P(x;t) + \tilde{\omega}\tilde{P}(t) \in \mathbb{P}^2,$$
(2.7)

where ω_0 , ω and $\tilde{\omega}$ are the (nonlinear) Z-type coefficients, see (2.3), associated to the regularity indicators I_0 , I and \tilde{I} , respectively for the polynomials P_{opt} , P and \tilde{P} , see (2.4).

Since $\tilde{I} = 0$, the global smoothness indicator τ in this case can be chosen as

$$\tau := |I - I_0|, \qquad (2.8)$$

The analysis in [47] shows that the reconstruction of Definition 2.2 achieves third order of accuracy for r = 1,2, provided that the exponent p in (2.3) is $p \ge 1$ and that $\epsilon = h^q$ for q = 1,2,3.

The reconstruction polynomial $R_j(x;t)$ given in (2.6) provides the approximation for each component u_j of \mathbf{u}_j for $x \in \Omega_j$. Thus, one can estimate the values $u(x_j \pm h/2, t)$, $t \ge 0$, with

$$u_{j+\frac{1}{2}}^{-}(t) = R_{j}\left(x_{j}+\frac{h}{2};t\right) \text{ and } u_{j+\frac{1}{2}}^{+}(t) = R_{j+1}\left(x_{j}+\frac{h}{2};t\right), \quad j=1,\cdots,N-1,$$
 (2.9a)

$$u_{\frac{1}{2}}^{-}(t) = u_{\text{out}}^{-} \text{ and } u_{\frac{1}{2}}^{+}(t) = R_{1}(a;t),$$
 (2.9b)

$$u_{N+\frac{1}{2}}^{-}(t) = R_N(b;t) \text{ and } u_{N+\frac{1}{2}}^{+}(t) = u_{\text{out}}^{+},$$
 (2.9c)

which are named *boundary extrapolated data* (BED). The outer values u_{out}^{\pm} are determined by the boundary conditions. For instance, for periodic boundary conditions the outer values u_{out}^{\pm} are set to the inner reconstructions at the last and first interfaces, respectively. Instead, for free-flow boundary conditions, the outer values u_{out}^{\pm} are set to the inner reconstructions at the first and last interfaces, respectively, namely imposing zero jumps at the boundaries of the physical domain.

Notice that at each interface the two BED in (2.9) are different, although computed at the same interface $x_{j+h/2}$. Therefore, in order to approximate the flux function at the interfaces, one introduces a consistent and monotone numerical flux function

$$(\mathbf{v},\mathbf{w}) \in \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathcal{F}(\mathbf{v},\mathbf{w}) \in \mathbb{R}^m,$$
 (2.10a)

such that

$$\mathbf{f}\left(\mathbf{u}(x_{j+\frac{1}{2}},t)\right) \approx \mathcal{F}\left(\mathbf{u}_{j+\frac{1}{2}}^{-}(t),\mathbf{u}_{j+\frac{1}{2}}^{+}(t)\right) \in \mathbb{R}^{m}.$$
(2.10b)

The function \mathcal{F} may be any approximate or exact Riemann solver, which is applied component-wise on vector-valued inputs. Finally, the exact system of ODEs (2.1) is reduced to a finite system of ODEs for the evolution of the cell averages. The right hand

side is completely defined by the space reconstruction along with the numerical flux, and one obtains

$$\frac{\mathrm{d}\mathbf{U}_{j}(t)}{\mathrm{d}t} = -\frac{1}{h} \left[\mathcal{F}_{j+\frac{1}{2}}(t) - \mathcal{F}_{j-\frac{1}{2}}(t) \right], \qquad (2.11)$$

which provides the approximation $\overline{\mathbf{U}}_{j}(t)$ of the cell averages $\overline{\mathbf{u}}_{j}(t)$ of the solution $\mathbf{u}(x,t)$, $x \in \Omega_{j}$, and where

$$\mathcal{F}_{j+\frac{1}{2}}(t) = \mathcal{F}\left(\mathbf{U}_{j+\frac{1}{2}}^{-}(t), \mathbf{U}_{j+\frac{1}{2}}^{+}(t)\right) \in \mathbb{R}^{m},$$
(2.12)

with $\mathbf{U}_{j+1/2}^{\pm}(t)$ BED of the data $\overline{\mathbf{U}}(t) = [\overline{\mathbf{U}}_1(t), \cdots, \overline{\mathbf{U}}_N(t)]^T$ according to (2.9).

2.2 Time integration: Third order Diagonally Implicit Runge-Kutta

In order to employ a timestep Δt which is not constrained by the CFL stability condition, we solve numerically equation (2.11) with a Diagonally Implicit Runge-Kutta (DIRK) scheme with *s* stages and general Butcher tableau

A typical assumption is that $c_k = \sum_{i=1}^{s} a_{ki}$, $k = 1, \dots, s$, and one has $\sum_{k=1}^{s} b_k = 1$ for consistency. Further, choosing a scheme in which a_{kk} is independent of k, the construction of the Jacobian in the nonlinear solvers is simplified. Clearly, one uses a Butcher tableau with order matching the order of the space reconstruction, in this case a third order accurate scheme. Therefore, DIRK schemes with number of stages $s \ge 2$ must be considered, e.g. see [3].

The space-time discretization leads to the fully-discrete scheme

$$\overline{\mathbf{U}}_{j}^{(k)} = \overline{\mathbf{U}}_{j}^{n} - \frac{\Delta t}{h} \sum_{i=1}^{k} a_{ki} \left[\mathcal{F}_{j+\frac{1}{2}}^{(i)} - \mathcal{F}_{j-\frac{1}{2}}^{(i)} \right], \quad k = 1, \cdots, s,$$
(2.14a)

$$\overline{\mathbf{U}}_{j}^{n+1} = \overline{\mathbf{U}}_{j}^{n} - \frac{\Delta t}{h} \sum_{k=1}^{s} b_{k} \left[\mathcal{F}_{j+\frac{1}{2}}^{(k)} - \mathcal{F}_{j-\frac{1}{2}}^{(k)} \right], \quad n \ge 0,$$
(2.14b)

$$\mathcal{F}_{j+\frac{1}{2}}^{(k)} = \mathcal{F}\left(\mathbf{U}_{j+\frac{1}{2}}^{-,(k)}, \mathbf{U}_{j+\frac{1}{2}}^{+,(k)}\right) \in \mathbb{R}^{m}$$
(2.14c)

for each $j = 1, \dots, N$, where Δt is the time-step, $\overline{\mathbf{U}}_{j}^{n} \approx \overline{\mathbf{u}}_{j}(n\Delta t)$, and $\mathbf{U}_{j+1/2}^{\mp,(k)}$ are the BED of the stage values $\overline{\mathfrak{U}}^{(k)} = [\overline{\mathbf{U}}_{1}^{(k)}, \dots, \overline{\mathbf{U}}_{N}^{(k)}]^{T}$, approximations of the solution at times $t^{(k)} =$

 $(n+c_k)\Delta t$, according to (2.9). Here and in the following, the notation suggests the use of a uniform time-step Δt . Nevertheless, the scheme can be formulated for a non-uniform time-step, but we prefer not to burden the notation.

The advantage of DIRK schemes is that the implicit computation of a given stage value (2.14a) can be performed sequentially from k = 1 to k = s. Therefore, at each time-step one has to solve *s* systems of nonlinear equations of size *mN* by *mN*:

$$\mathbf{G}\left(\overline{\mathfrak{U}}^{(k)}\right) := \overline{\mathfrak{U}}^{(k)} + \frac{a_{kk}\Delta t}{h} \Delta \mathfrak{F}^{(k)} - \overline{\mathfrak{U}}^n + \frac{\Delta t}{h} \sum_{i=1}^{k-1} a_{ki} \Delta \mathfrak{F}^{(i)} = \mathbf{0}, \qquad (2.15)$$

where $\Delta \mathfrak{F}^{(k)} \in \mathbb{R}^{mN}$ whose *j*-th block is $\Delta \mathfrak{F}_{j}^{(k)} = (\mathcal{F}_{j+1/2}^{(k)} - \mathcal{F}_{j-1/2}^{(k)}) \in \mathbb{R}^{m}$, for $j=1,\dots,N$. Above we have highlighted the term $\Delta \mathfrak{F}^{(k)}$, which makes the system for the *k*-th stage value nonlinear. In fact, the other flux differences $\Delta \mathfrak{F}^{(i)}$, $i = 1, \dots, k-1$, are already available, thanks to the structure of DIRK schemes.

As already noticed in [43], **G** has two sources of nonlinearity. One is unavoidable because it is due to the physics when the phenomenon under study is described by the nonlinear flux function **f** in (1.1). The second, instead, is introduced by the high order space reconstruction procedure, needed for the computation of the BED, which is highly nonlinear because of the nonlinear weights (2.3) and of the regularity indicators (2.4). Therefore, even for linear PDEs, a standard implicit scheme requires a nonlinear solver to find the solution of $\mathbf{G}(\overline{\mathfrak{U}}^{(k)}) = \mathbf{0}$, in equation (2.15) for $k = 1, \dots, s$. Typically, one uses Newton-Raphson's method, which requires assembling the Jacobian of the nonlinear system **G**, resulting in a high computational cost. In fact, the Jacobian required for the Newton iterations has bandwidth one point larger than the stencil size in each direction, so for a third order scheme it is a block-pentadiagonal matrix with $m \times m$ blocks, with entries depending on the nonlinear weights and on the regularity indicators, which have very complicated expressions.

The Quinpi approach, introduced in [43] for scalar conservation laws, provides a way to circumvent the nonlinearity determined by the high order reconstruction procedure. In the following section, we extend the Quinpi idea to the hyperbolic system (1.1).

3 Third order Quinpi scheme for one-dimensional hyperbolic systems

The name Quinpi stands for implicit CWENO and it is based on a predictor-corrector approach to avoid the nonlinearity of the high order scheme, introduced by the space reconstruction, and keep the nonlinearity of the flux function **f** only.

The Quinpi idea relies on the following considerations on the structure of most essentially-non-oscillatory reconstructions. Since reconstructions are typically applied component-wise, we use the notation for scalar conservation laws in this section. Let $S_p = \{\Omega_{j-p}, \dots, \Omega_{j+p}\}$ be a stencil of 2p+1 computational cells, and $P \in \mathbb{P}^{2p}$ be the optimal interpolating polynomial of the cell averages in S_p . Then, the dependence of P on the data is linear, i.e.

$$P(x;t) = \sum_{\alpha=-p}^{p} \mu_{j,\alpha}(x) \overline{u}_{j+\alpha}(t).$$

Therefore, unrolling the linearity with respect to the data of all the interpolating polynomials involved in the third order reconstruction procedure of Definition 2.2, one can write the reconstruction polynomial as

$$\begin{split} R_{j}(x;t) &= \sum_{i=0,L,R} \omega_{i} \left(\{ \overline{u}_{k}(t) \}_{k \in \mathcal{S}_{opt}} \right) P_{j}^{i}(x) \\ &= \sum_{i=0,L,R} \omega_{i} \left(\{ \overline{u}_{k}(t) \}_{k \in \mathcal{S}_{opt}} \right) \sum_{\alpha=-1}^{1} \mu_{j,\alpha}^{i}(x) \overline{u}_{j+\alpha+\delta_{j1}-\delta_{jN}}(t) \\ &= \sum_{\alpha=-1}^{1} W_{j,\alpha} \left(x; \{ \overline{u}_{k}(t) \}_{k \in \mathcal{S}_{opt}} \right) \overline{u}_{j+\alpha+\delta_{j1}-\delta_{jN}}(t), \end{split}$$

where we have collected the nonlinear weights in the quantities

$$W_{j,\alpha}\left(x;\{\overline{u}_{k}(t)\}_{k\in\mathcal{S}_{opt}}\right) = \sum_{i=0,L,R} \omega_{i}\left(\{\overline{u}_{k}(t)\}_{k\in\mathcal{S}_{opt}}\right) \mu_{j,\alpha}^{i}(x),$$

and δ_{ij} denotes the Kronecker delta. For the first and the last cell, the range of summation over α is adjusted according to the stencil of the reconstruction. Then, it is possible to write the inner BED in (2.9) as

$$u_{j+\frac{1}{2}}^{-}(t) = R_{j}\left(x_{j+\frac{1}{2}};t\right)$$

$$= \sum_{\alpha=-1}^{1} W_{j,\alpha}\left(x_{j+\frac{1}{2}};\{\overline{u}_{k}(t)\}_{k\in\mathcal{S}_{opt}}\right)\overline{u}_{j+\alpha+\delta_{j1}-\delta_{jN}}(t),$$

$$u_{j+\frac{1}{2}}^{+}(t) = R_{j+1}\left(x_{j+\frac{1}{2}};t\right)$$

$$= \sum_{\alpha=-1}^{1} W_{j+1,\alpha}\left(x_{j+\frac{1}{2}};\{\overline{u}_{k}(t)\}_{k\in\mathcal{S}_{opt}}\right)\overline{u}_{j+1+\alpha+\delta_{j1}-\delta_{jN}}(t).$$
(3.1)

We have highlighted the dependence of $W_{j,\alpha}$ on the data $\{\overline{u}_k\}_{k\in S_{opt}}$, since this is highly nonlinear because it contains the nonlinear weights.

Remark 3.1. For the third order reconstruction considered in this work, the quantities $W_{j,\alpha}(x; \{\overline{u}_k(t)\}_{k \in S_{opt}})$ have the following expressions. For the left BED at $x = x_{j+1/2}$, with $j = 2, \dots, N-1$:

$$\begin{split} W_{j,-1} &= \left(\frac{-1+3d_L}{6}\right) \frac{\omega_0}{d_0} - \frac{1}{2}\omega_L, \quad W_{j,0} = \left(\frac{5-9d_L - 3d_R}{6}\right) \frac{\omega_0}{d_0} + \frac{3}{2}\omega_L + \frac{1}{2}\omega_R, \\ W_{j,1} &= \left(\frac{2-3d_R}{6}\right) \frac{\omega_0}{d_0} + \frac{1}{2}\omega_R. \end{split}$$

For the right BED at $x = x_{j+1/2}$, with $j = 1, \dots, N-2$:

$$\begin{split} W_{j+1,-1} &= \left(\frac{2-3d_L}{6}\right)\frac{\omega_0}{d_0} + \frac{1}{2}\omega_L, \quad W_{j+1,0} = \left(\frac{5-3d_L - 9d_R}{6}\right)\frac{\omega_0}{d_0} + \frac{1}{2}\omega_L + \frac{3}{2}\omega_R \\ W_{j+1,1} &= \left(\frac{-1+3d_R}{6}\right)\frac{\omega_0}{d_0} - \frac{1}{2}\omega_R. \end{split}$$

The main idea of Quinpi is to exploit a predictor $\{\overline{u}_{j}^{\star}(t)\}_{j=1}^{N}$ of the solution $\{\overline{u}_{j}(t)\}_{j=1}^{N}$ at time *t* and use it to pre-compute and freeze the nonlinear weights in (3.1), so that the BED can be approximated as

$$u_{j+\frac{1}{2}}^{-}(t) \approx \hat{u}_{j+\frac{1}{2}}^{-}(t) := \sum_{\alpha=-1}^{1} W_{j,\alpha} \left(x_{j+\frac{1}{2}}; \{ \overline{u}_{k}^{\star}(t) \}_{k \in \mathcal{S}_{opt}} \right) \overline{u}_{j+\alpha+\delta_{j1}-\delta_{jN}}(t),$$

$$u_{j+\frac{1}{2}}^{+}(t) \approx \hat{u}_{j+\frac{1}{2}}^{+}(t) := \sum_{\alpha=-1}^{1} W_{j+1,\alpha} \left(x_{j+\frac{1}{2}}; \{ \overline{u}_{k}^{\star}(t) \}_{k \in \mathcal{S}_{opt}} \right) \overline{u}_{j+1+\alpha+\delta_{j1}-\delta_{jN}}(t).$$
(3.2)

In this way, the complete scheme would be linear with respect to the space reconstruction, and nonlinear only through the flux function.

The proposal of a predictor to linearize an implicit high order scheme was first discussed in [31] for WENO reconstructions. There, the solution of a first order explicit scheme is used as predictor in order to compute the nonlinear WENO weights. Instead, in [4,43] an implicit first order scheme was chosen as predictor. Although an implicit predictor is more expensive than the explicit approach proposed in [31], it is stable and non-oscillatory even for higher Courant numbers, thus allowing for reliable prediction of the nonlinear weights.

In this work, we follow the approach in [43] and describe it for systems of conservation laws below. With respect to the above description, in the case of a system of *m* conservation laws, one has a reconstruction polynomial $\mathbf{R}_j(x;t) \in (\mathbb{P}^{2p})^m$ computed componentwise, *m*-vectors $\boldsymbol{\omega}_i$ of nonlinear coefficients in each cell computed component-wise, while of course the linear coefficients $\mu_{j,\alpha}^i$ are independent of the component being reconstructed.

3.1 The space-time first order implicit prediction

Without loss of generality, assume that the nodes c_1, \dots, c_s of the DIRK method are ordered. Then, we approximate the system of ODEs (2.11) with an implicit first order scheme at any time $t^{(k)} = (n+c_k)\Delta t \in [n\Delta t, (n+1)\Delta t]$, where Δt is the time-step of the high order scheme (2.14). Specifically, the system is numerically approximated in space using piecewise constant reconstructions and integrated in time using a composite backward Euler method, providing the *s* approximations $\overline{\mathfrak{U}}^{\star,(k)} = [\overline{\mathfrak{U}}_1^{\star,(k)}, \cdots, \overline{\mathfrak{U}}_N^{\star,(k)}]^T$, $k=1,\dots,s$.

Therefore, for each $j = 1, \dots, N$, $\overline{\mathbf{U}}_{j}^{\star,(k)} \approx \overline{\mathbf{u}}_{j}(t^{(k)})$ is given by

$$\overline{\mathbf{U}}_{j}^{\star,(k)} = \overline{\mathbf{U}}_{j}^{\star,(k-1)} - \frac{\theta_{k}\Delta t}{h} \left[\mathcal{F}_{j+\frac{1}{2}}^{\star,(k)} - \mathcal{F}_{j-\frac{1}{2}}^{\star,(k)} \right], \quad k = 1, \cdots, s,$$
(3.3a)

$$\overline{\mathbf{U}}_{j}^{\star,n+1} = \overline{\mathbf{U}}_{j}^{n} - \frac{\Delta t}{h} \sum_{k=1}^{s} \theta_{k} \left[\mathcal{F}_{j+\frac{1}{2}}^{\star,(k)} - \mathcal{F}_{j-\frac{1}{2}}^{\star,(k)} \right], \quad n \ge 0,$$
(3.3b)

$$\mathcal{F}_{j+\frac{1}{2}}^{\star,(k)} = \mathcal{F}\left(\overline{\mathbf{U}}_{j}^{\star,(k)}, \overline{\mathbf{U}}_{j+1}^{\star,(k)}\right) \in \mathbb{R}^{m}, \tag{3.3c}$$

where $\theta_k := c_k - c_{k-1}$, with $c_0 = 0$ and $\overline{\mathbf{U}}_j^{\star,(0)} := \overline{\mathbf{U}}_j^n$, and where the convention

$$\overline{\mathbf{U}}_0^{\star,(k)} = \! u_{\mathsf{out}}^-, \quad \overline{\mathbf{U}}_{N+1}^{\star,(k)} = \! u_{\mathsf{out}}^+,$$

is used. The outer values u_{out}^{\mp} are imposed by the chosen boundary conditions. For instance, for periodic conditions one has

$$\overline{\mathbf{U}}_{0}^{\star,(k)} = \overline{\mathbf{U}}_{N}^{\star,(k)}, \quad \overline{\mathbf{U}}_{N+1}^{\star,(k)} = \overline{\mathbf{U}}_{1}^{\star,(k)}, \quad (3.4)$$

whereas for free-flow conditions one gets

$$\overline{\mathbf{U}}_{0}^{\star,(k)} = \overline{\mathbf{U}}_{1}^{\star,(k)}, \quad \overline{\mathbf{U}}_{N+1}^{\star,(k)} = \overline{\mathbf{U}}_{N}^{\star,(k)}.$$
(3.5)

For more general boundary conditions, one should project on characteristic variables, providing the prescribed upwind conditions. Summarizing, the first order scheme (3.3) is equivalent to applying a DIRK scheme with Butcher tableau given by

where the coefficients c_i , $i = 1, \dots, s$, are the same as the high order DIRK in (2.13).

Notice that the numerical flux function \mathcal{F} in (3.3c) is now computed on piecewise constant, unlimited, reconstructions from the cell averages. In fact, first order schemes do not require space-limiting, because they are unconditionally Total Variation Diminishing, see [43, Section 2]. Therefore, despite of the high order approximation (2.14), the first order scheme (3.3) is characterized by a single nonlinearity, that is the one induced by the flux function **f**. Computing (3.3a) requires the solution of *s* nonlinear systems of dimension *mN*, which are fully linear with respect to the unknown cell averages:

$$\mathbf{G}^{\star}(\overline{\mathfrak{U}}^{\star,(k)}) := \overline{\mathfrak{U}}^{\star,(k)} + \frac{\theta_k \Delta t}{h} \Delta \mathfrak{F}^{\star,(k)} - \overline{\mathfrak{U}}^{\star,(k-1)} = \mathbf{0}, \qquad (3.7)$$

where $\Delta \mathfrak{F}^{\star,(k)} \in \mathbb{R}^{mN}$ whose *j*-th block is $\Delta \mathfrak{F}_{j}^{\star,(k)} = (\mathcal{F}_{j+1/2}^{\star,(k)} - \mathcal{F}_{j-1/2}^{\star,(k)}) \in \mathbb{R}^{m}$, for $j = 1, \dots, N$.

The solution of (3.7) requires the use of a nonlinear solver *s* times within a single time-step. In this work, we rely on Newton's iterations

$$\overline{\mathfrak{U}}_{(\ell+1)}^{\star,(k)} = \overline{\mathfrak{U}}_{(\ell)}^{\star,(k)} - \left(\mathbb{I} + \frac{\theta_k \Delta t}{h} J^{\star}\left(\overline{\mathfrak{U}}_{(\ell)}^{\star,(k)}\right)\right)^{-1} \mathbf{G}^{\star}\left(\overline{\mathfrak{U}}_{(\ell)}^{\star,(k)}\right), \quad \ell \ge 0,$$
(3.8)

with given initial guess $\overline{\mathfrak{U}}_{(0)}^{\star,(k)} = \overline{\mathfrak{U}}^{\star,(k-1)}$. Here, $\mathbb{I} \in \mathbb{R}^{mN \times mN}$ is the identity matrix, whereas J^{\star} is the Jacobian matrix of the numerical flux difference $\Delta \mathfrak{F}^{\star}$ which is organized in $N \times N$ blocks of size $m \times m$. Then, each block of the Jacobian is given by

$$(J^{\star})_{ji} = \frac{\partial \Delta \mathfrak{F}_{j}^{\star}}{\partial \overline{\mathbf{U}}_{i}^{\star}} \in \mathbb{R}^{m \times m}, \quad j, i = 1, \cdots, N.$$

In particular, for a first order space approximation one has that the Jacobian is a tridiagonal block matrix of the form

$$\frac{\partial \Delta \mathfrak{F}_{j}^{\star}}{\partial \overline{\mathbf{U}}_{i}^{\star}} = \begin{cases} -\frac{\partial \mathcal{F}}{\partial \mathbf{v}} \left(\overline{\mathbf{U}}_{i}^{\star}, \overline{\mathbf{U}}_{i+1}^{\star} \right), & \text{if } i = j - 1, \\ \frac{\partial \mathcal{F}}{\partial \mathbf{v}} \left(\overline{\mathbf{U}}_{i}^{\star}, \overline{\mathbf{U}}_{i+1}^{\star} \right) - \frac{\partial \mathcal{F}}{\partial \mathbf{w}} \left(\overline{\mathbf{U}}_{i-1}^{\star}, \overline{\mathbf{U}}_{i}^{\star} \right), & \text{if } i = j, \\ \frac{\partial \mathcal{F}}{\partial \mathbf{w}} \left(\overline{\mathbf{U}}_{i-1}^{\star}, \overline{\mathbf{U}}_{i}^{\star} \right), & \text{if } i = j + 1, \end{cases}$$

for $j=2, \dots, N-1$, where $\partial_{\mathbf{v}} \mathcal{F}$ and $\partial_{\mathbf{w}} \mathcal{F}$ denote the Jacobian of the vector-valued numerical flux function (2.10) with respect to its first and second variable, respectively.

3.2 The space-time third order implicit correction

Assume that the first order predictor is available at a time $t^{(k)} \in [n\Delta t, (n+1)\Delta t]$. The obtained approximation $\overline{\mathfrak{U}}^{\star,(k)}$ is used to evaluate the nonlinear terms of the high order BED as explained at the beginning of this section, see (3.2). Therefore, the third order "correction" at time $t^{(k)}$ can be computed as in (2.14a) with

$$\mathcal{F}_{j+\frac{1}{2}}^{(k)} = \mathcal{F}\left(\mathbf{U}_{j+\frac{1}{2}}^{-,(k)}, \mathbf{U}_{j+\frac{1}{2}}^{+,(k)}\right) \approx \hat{\mathcal{F}}_{j+\frac{1}{2}}^{(k)} = \mathcal{F}\left(\hat{\mathbf{U}}_{j+\frac{1}{2}}^{-,(k)}, \hat{\mathbf{U}}_{j+\frac{1}{2}}^{+,(k)}\right),$$
(3.9)

where $\hat{\mathbf{U}}_{j+\frac{1}{2}}^{\mp,(k)}$ are the linearized BED, which are fully linear with respect to the unknown cell averages at time $t^{(k)} = t_n + c_k \Delta t$. Therefore, the nonlinear system

$$\mathbf{G}\left(\overline{\mathfrak{U}}^{(k)}\right) := \overline{\mathfrak{U}}^{(k)} + \frac{a_{kk}\Delta t}{h} \Delta \widehat{\mathfrak{F}}^{(k)} - \overline{\mathfrak{U}}^{(k-1)} = \mathbf{0}, \qquad (3.10)$$

obtained from (2.14a) and (3.9), can be tackled with Newton's iterations

$$\overline{\mathfrak{U}}_{(\ell+1)}^{(k)} = \overline{\mathfrak{U}}_{(\ell)}^{(k)} - \left(\mathbb{I} + \frac{a_{kk}\Delta t}{h} J\left(\overline{\mathfrak{U}}_{(\ell)}^{(k)}\right)\right)^{-1} \mathbf{G}\left(\overline{\mathfrak{U}}_{(\ell)}^{(k)}\right), \quad \ell \ge 0,$$
(3.11)

with initial guess $\overline{\mathfrak{U}}_{(0)}^{(k)}$ set to the predictor's output. The matrix $J \in \mathbb{R}^{mN \times mN}$ is the Jacobian of the system (3.10) and one has that each block $(J)_{ji} \in \mathbb{R}^{m \times m}$, $j, i = 1, \dots, N$, can be written as

$$(J)_{ji} = \frac{\partial \Delta \hat{\mathfrak{F}}_{j}}{\partial \overline{\mathbf{U}}_{i}} = \frac{\partial \mathcal{F}}{\partial \mathbf{v}} \left(\hat{\mathbf{U}}_{j+\frac{1}{2}}^{-}, \hat{\mathbf{U}}_{j+\frac{1}{2}}^{+} \right) \frac{\partial \overline{\mathbf{U}}_{j+\frac{1}{2}}^{-}}{\partial \overline{\mathbf{U}}_{i}} + \frac{\partial \mathcal{F}}{\partial \mathbf{w}} \left(\hat{\mathbf{U}}_{j+\frac{1}{2}}^{-}, \hat{\mathbf{U}}_{j+\frac{1}{2}}^{+} \right) \frac{\partial \overline{\mathbf{U}}_{j+\frac{1}{2}}^{+}}{\partial \overline{\mathbf{U}}_{i}} - \frac{\partial \mathcal{F}}{\partial \mathbf{v}} \left(\hat{\mathbf{U}}_{j-\frac{1}{2}}^{-}, \hat{\mathbf{U}}_{j+\frac{1}{2}}^{+} \right) \frac{\partial \hat{\mathbf{U}}_{j+\frac{1}{2}}^{+}}{\partial \overline{\mathbf{U}}_{i}} - \frac{\partial \mathcal{F}}{\partial \overline{\mathbf{U}}_{i}} \left(\hat{\mathbf{U}}_{j-\frac{1}{2}}^{-}, \hat{\mathbf{U}}_{j+\frac{1}{2}}^{+} \right) \frac{\partial \hat{\mathbf{U}}_{j+\frac{1}{2}}^{+}}{\partial \overline{\mathbf{U}}_{i}} - \frac{\partial \mathcal{F}}{\partial \overline{\mathbf{U}}_{i}} \left(\hat{\mathbf{U}}_{j-\frac{1}{2}}^{-}, \hat{\mathbf{U}}_{j+\frac{1}{2}}^{+} \right) \frac{\partial \hat{\mathbf{U}}_{j+\frac{1}{2}}^{+}}{\partial \overline{\mathbf{U}}_{i}}$$

where $\partial_{\mathbf{v}}\mathcal{F}$ and $\partial_{\mathbf{w}}\mathcal{F}$ contain the Jacobian of the flux function **f**, whereas $\partial_{\overline{\mathbf{U}}_i}\mathbf{U}_{j\pm^{1/2}}^{\mp}$ are assembled with the frozen nonlinear weights, cf. (3.2). Observe that the width of the band of the block-banded matrix that defines the Jacobian *J* is still as in the fully implicit approach, because of the stencil of the space reconstructions, but the entries can now be computed explicitly.

For nontrivial boundary conditions, the above expression for the Jacobian blocks has to be modified according to the dependence of the external BED ($\overline{\mathbf{U}}_{1/2}^-$ and $\overline{\mathbf{U}}_{N+1/2}^+$) on the interior data.

Finally, once all the stage values $\overline{\mathfrak{U}}^{(k)}$, $k=1,\dots,s$, are computed, the third order solution $\overline{\mathfrak{U}}^{n+1}$ at time level $(n+1)\Delta t$ is obtained as in (2.14b).

Remark 3.2 (Implicit reconstruction along characteristic variables.). The Quinpi framework can be used also for reconstruction along characteristic variables. The linearity of the scheme can be kept provided that the nonlinear transformation matrix from conservative to characteristic variables is computed by the predictor scheme and stored at each time-step $t^{(k)} = t^n + c_k \Delta t$, $k = 1, \dots, s$.

More precisely, let **U** denote the vector of conservative variables and let **V** denote the vector of characteristic variables which are linked through the nonlinear transformation

$$\mathbf{V} = T(\mathbf{U}).$$

Then, the time advancement $\overline{\mathbf{U}}_{j}^{(k-1)} \mapsto \overline{\mathbf{U}}_{j}^{(k)}$ on the cell Ω_{j} , with reconstruction from cell averages performed in characteristic variables, is obtained for $k = 2, \dots, s$ as follows.

First one can perform the time advancement of the predictor solution $\overline{\mathbf{U}}_{j}^{\star}$ and store the nonlinear transformation matrix:

$$\{\overline{\mathbf{U}}_{j}^{\star,(k-1)}\} \xrightarrow{\text{advance}} \{\overline{\mathbf{U}}_{j}^{\star,(k)}\} \xrightarrow{\text{store}} \{T_{j}^{(k)}\},$$

where $T_j^{(k)}$ is the map to characteristic variables on the state $\overline{\mathbf{U}}_j^{\star,(k)}$, i.e. the map whose columns are the right eigenvectors of the flux Jacobian at $\overline{\mathbf{U}}_i^{\star,(k)}$.

Then, the high-order solution at time $t^{(k)}$ is obtained as

$$\{\overline{\mathbf{U}}_{\ell}^{(k-1)}, \overline{\mathbf{U}}_{\ell}^{\star, (k)}\}_{\ell \in \mathcal{S}_{\text{opt}}} \xrightarrow{T_{j}^{(k)}, \text{ linear. BED}} \{\widehat{\mathbf{V}}_{j \pm \frac{1}{2}}^{(k)}\} \xrightarrow{(T_{j}^{(k)})^{-1}} \{\widehat{\mathbf{U}}_{j \pm \frac{1}{2}}^{(k)}\} \xrightarrow{\text{advance}} \{\overline{\mathbf{U}}_{j}^{(k)}\}.$$

3.2.1 On the accuracy of the reconstruction in the smooth case

In CWENO reconstructions optimal accuracy is obtained provided that the nonlinear weights ω_k are sufficiently close to the linear weights d_k . More precisely, for the third order CWENO scheme a sufficient condition to have optimal accuracy on smooth data is, cf. [19],

$$d_k - \omega_k = \mathcal{O}(h), \quad k = 0, L, R$$

Indeed, let $R_j^{\text{CWZ}} \in \mathbb{P}^2$ be the reconstruction polynomial, defined in (2.2), of a sufficiently smooth function *u*. Then, the reconstruction error is

$$u(x,\cdot) - R_{j}^{\text{CWZ}}(x;\cdot) = \underbrace{(u(x,\cdot) - P_{\text{opt}}(x;\cdot))}_{\mathcal{O}(h^{3})} + \sum_{k=L,R} (d_{k} - \omega_{k}) \underbrace{(P_{k}(x;\cdot) - u(x,\cdot))}_{\mathcal{O}(h^{2})}, \quad x \in \Omega_{j}, \quad (3.12)$$

when the polynomials P_{opt} , P_L and P_R interpolate the exact cell averages of u. Therefore, the optimal order of accuracy, in this case 3, is obtained on smooth data if

$$d_k - \omega_k = \mathcal{O}(h), \quad k = 0, L, R. \tag{3.13}$$

We investigate numerically the convergence rate $d_k - \omega_k$ when the nonlinear weights are computed on a low order approximation of the cell-averages of a smooth function *u*. Thus, let us consider

$$u(x) = \sin(\pi x) + \sin(15\pi x)\exp(-20x^2), x \in [-1,1].$$

We discretize the space domain by *N* cells with periodic boundary conditions. We take the optimal weights $d_0 = 0.75$, $d_L = d_R = 0.125$. Then, we compute the nonlinear weights $\hat{\omega}_{j,k}$, k = 0, L, R, based on a first order approximation of the cell averages of *u* in the *j*-th cell, and the nonlinear weights $\omega_{j,k}$, k = 0,1,2, on a third order approximation of the cell averages of *u* in the *j*-th cell. We study the behavior of the errors

$$\widehat{\operatorname{err}}(N) = \frac{1}{N} \sum_{j=1}^{N} \max_{k=0,L,R} |d_k - \hat{\omega}_{j,k}|,$$
$$\operatorname{err}(N) = \frac{1}{N} \sum_{j=1}^{N} \max_{k=0,L,R} |d_k - \omega_{j,k}|,$$

Ν	êrr	rate	err	rate
20	3.58e - 1	-	3.83e - 1	-
40	2.56e - 1	0.48	2.92e - 1	0.39
80	2.43e - 1	0.08	2.46e - 1	0.25
160	1.57e - 1	0.63	1.56e - 1	0.65
320	7.23e - 2	1.12	7.23e - 2	1.11
640	3.96e - 2	0.87	3.96e - 2	0.87
1,280	1.76e - 2	1.17	1.76e - 2	1.17
2,560	5.59e - 3	1.65	5.59e - 3	1.65
5,120	1.37e - 3	2.03	1.37e - 3	2.03

Table 1: Experimental convergence of the nonlinear to the linear weights.

under grid refinement. The results are given in Table 1 which shows that the sufficient condition on the convergence rate of the nonlinear weights to the linear weights to guarantee optimal accuracy is fulfilled in both cases. In Quinpi the accuracy conditions in (3.12) are guaranteed since the P_{opt} and P_k polynomials are computed on the high order cell averages in the nonlinear solver, while the first order predictor is accurate enough to guarantee condition (3.13) on the nonlinear weights.

3.3 Conservative a-posteriori time-limiting based on numerical entropy

The solution obtained with the predictor-corrector approach is third order accurate, and has some control over spurious oscillations thanks to the limited CWENO space reconstruction. However, when using a large Δt , space limiting is not enough, and the high order solution may still exhibit oscillations as already noticed in [4, 24, 43]. In such situations, limiting in time is also required. For this reason, the Quinpi scheme that we propose in this paper employs a time-limiting procedure of the third order solution.

In [43] the time-limiting is performed in a WENO-like framework. In fact, nonlinear weights are suitably defined in order to blend cell averages between the computed high order solution and the low order predictor, which is reliable, stable and non-oscillatory. However, the procedure in [43] does not preserve mass conservation property and must be followed by a suitable conservative correction, analogously to [8] in adaptive mesh refinement.

Extending the conservative correction of [43] to a generic system (1.1) is nontrivial. Instead, conservation can be ensured avoiding a cell-centered blending by means of flux-based Runge-Kutta, as, e.g., in [4,35]. Similarly, a flux-centered conservative aposteriori time-limiting inspired by the MOOD technique [13,14] was investigated in [50]. MOOD was originally designed as an a-posteriori space-limiting technique for multidimensional finite volume schemes. Instead, in [50], the typical MOOD detectors were used to limit the high order solution at time level $(n+1)\Delta t$. However, contrary to the classical MOOD approach, in [50] the method uses a convex combination of the high order and the low-order numerical fluxes at the interfaces of oscillatory cells. A similar idea was also employed in [26] for a-posteriori limiting of fully implicit finite volume schemes on transport networks.

In this paper we still rely on the MOOD technique, and the high order numerical fluxes of oscillatory cells are dropped to the low-order numerical fluxes of the predictor. The novelty here is that the detection of troubled cells is performed with the numerical entropy production introduced in [40,41], extended to balance laws in [42], and already exploited in [44,45] as a-posteriori error of adaptive schemes and indicator of the qualitative structure of the flow.

3.3.1 Indicators based on numerical entropy production

Assume that system (1.1) is endowed with an entropy-entropy flux pair, namely that there exists a convex function $\mathbf{u} \in \mathbb{R}^m \mapsto \eta(\mathbf{u}) \in \mathbb{R}$ and a corresponding entropy flux $\mathbf{u} \in \mathbb{R}^m \mapsto \psi(\mathbf{u}) \in \mathbb{R}$ such that $\nabla^T \eta(\mathbf{u}) \mathbf{f}'(\mathbf{u}) = \nabla^T \psi(\mathbf{u})$. Then, all entropy solutions of (1.1) satisfy the entropy inequality

$$\frac{\partial}{\partial t}\eta(\mathbf{u}(x,t)) + \frac{\partial}{\partial x}\psi(\mathbf{u}(x,t)) \le 0, \tag{3.14}$$

in the weak sense. It is well known that (3.14) is an equality on smooth flows. Instead, if the solution has a singularity, the sign of (3.14) selects the unique physically admissible weak solution of (1.1). For this reason, the novel idea of [40] was to consider the numerical residual of the entropy inequality as an a-posteriori error indicator for the numerical solution of (1.1), since it is a scalar value (even in the case of systems of conservation laws) which provides information on the size of the local truncation error and on the presence of singularities. Recently, the numerical entropy residual has been employed also in the context of stochastic Galerkin formulations of hyperbolic systems, see [29].

Following [41] we give the subsequent definition.

Definition 3.1. Let $\overline{\mathbf{U}}^n$ be the solution of (2.11) obtained using a *s*-stage Runge-Kutta scheme of order *p* with weights $\{b_i\}_{i=1}^s$ and time-step Δt . Then, the numerical entropy production $(S^p)_j^n$ of the scheme in the control volume $V_j^n = \Omega_j \times [n\Delta t, (n+1)\Delta t]$ is given by

$$(S^{p})_{j}^{n} = \frac{1}{\Delta t} \left[\overline{Q} \left(\eta \left(\overline{\mathbf{U}}^{n+1} \right) \right)_{j} - \overline{Q} \left(\eta \left(\overline{\mathbf{U}}^{n} \right) \right)_{j} + \frac{\Delta t}{h} \sum_{i=1}^{s} b_{i} \left(\Psi_{j+\frac{1}{2}}^{(i)} - \Psi_{j-\frac{1}{2}}^{(i)} \right) \right],$$
(3.15)

where

$$\Psi_{j+\frac{1}{2}}^{(i)} = \Psi\left(\mathbf{U}_{j+\frac{1}{2}}^{-,(i)}, \mathbf{U}_{j+\frac{1}{2}}^{+,(i)}\right)$$

is a numerical entropy flux function consistent with the exact entropy flux ψ of (3.14), with $(\mathbf{v}, \mathbf{w}) \in \mathbb{R}^m \times \mathbb{R}^m \to \Psi(\mathbf{v}, \mathbf{w}) \in \mathbb{R}$, and where $\overline{\mathbf{U}}_{j\pm 1/2}^{\mp,(i)}$ are the BED of the numerical solution. In (3.15), $\overline{Q}(\cdot)_j$ denotes a quadrature rule of order p to compute the cell average on the cell Ω_j .

The rate of convergence of (3.15) is shown in [41]. In particular, for a general numerical scheme of order p one has

$$(S^{p})_{j}^{n} = \begin{cases} \mathcal{O}(h^{p}), & \text{if the solution is regular on } \Omega_{j}, \\ \mathcal{O}(h^{-1}), & \text{if the solution has a shock on } \Omega_{j}, \\ \mathcal{O}(h), & \text{if } \Omega_{j} \text{ exhibits a rarefaction corner}, \\ \mathcal{O}(1), & \text{if } \Omega_{j} \text{ exhibits a contact wave.} \end{cases}$$

In other words, $(S^p)_j^n \to 0$, as $h \to 0$, with the same rate of the local truncation error of the scheme on smooth flows. Whereas, the numerical entropy production increases in magnitude on shocks.

For the first order predictor solution $\overline{\mathfrak{U}}^*$, defined in Section 3.1, we can consider the midpoint quadrature rule and observe that the cell average and the pointwise value at the cell center are $\mathcal{O}(h^2)$ apart, and thus set $\overline{\mathcal{Q}}(\eta(\overline{\mathbf{U}}^n))_j = \eta(\overline{\mathbf{U}}_j^n)$, in (3.15). The numerical entropy production becomes

$$\left(S^{1}\right)_{j}^{n} = \frac{1}{\Delta t} \left[\eta \left(\overline{\mathbf{U}}_{j}^{\star,n+1}\right) - \eta \left(\overline{\mathbf{U}}_{j}^{\star,n}\right) + \frac{\Delta t}{h} \sum_{i=1}^{s} \theta_{i} \left(\Psi_{j+\frac{1}{2}}^{\star,(i)} - \Psi_{j-\frac{1}{2}}^{\star,(i)}\right) \right], \quad (3.16)$$

where

$$\Psi_{j+\frac{1}{2}}^{\star,(i)} = \Psi\left(\mathbf{U}_{j}^{\star,(i)}, \mathbf{U}_{j+1}^{\star,(i)}\right).$$

Instead, for the third order solution $\overline{\mathfrak{U}}$ defined in Section 3.2, in this work we consider the two-point Gauss-Legendre quadrature rule:

$$\overline{Q}\left(\eta\left(\overline{\mathbf{U}}^{n}\right)\right)_{j}=\frac{1}{2}\left(\eta\left(\mathbf{U}_{j-\frac{\sqrt{3}}{6}}^{n}\right)+\eta\left(\mathbf{U}_{j+\frac{\sqrt{3}}{6}}^{n}\right)\right),$$

where $\mathbf{U}_{j\pm\sqrt{3}/6}^{n}$ denote the reconstructions of the solution in $x_{j}\pm\sqrt{3}h/6$. Observe that the computation of the quadrature rule does not require significant additional computational cost since the computation of $\mathbf{U}_{j\pm\sqrt{3}/6}^{n}$ involves just the evaluation of a CWENO reconstruction polynomial, which is already available, because it is computed in order to advance the solution from $n\Delta t$ to $(n+1)\Delta t$. Thus, for the third order solution, the numerical entropy production becomes

$$(S^{3})_{j}^{n} = \frac{1}{\Delta t} \left[\frac{1}{2} \left(\eta \left(\mathbf{U}_{j-\frac{\sqrt{3}}{6}}^{n+1} \right) + \eta \left(\mathbf{U}_{j+\frac{\sqrt{3}}{6}}^{n+1} \right) \right) - \frac{1}{2} \left(\eta \left(\mathbf{U}_{j-\frac{\sqrt{3}}{6}}^{n} \right) + \eta \left(\mathbf{U}_{j+\frac{\sqrt{3}}{6}}^{n} \right) \right) + \frac{\Delta t}{h} \sum_{i=1}^{s} b_{i} \left(\Psi_{j+\frac{1}{2}}^{(i)} - \Psi_{j-\frac{1}{2}}^{(i)} \right) \right].$$

$$(3.17)$$

At the discrete level, $(S^1)_j^n$ and $(S^3)_j^n$ provide a regularity indicator of the predictor and of the corrector solution, respectively. Therefore, in order to determine where time-limiting has to occur, we detect spurious oscillations using $(S^1)_j^n$ and $(S^3)_j^n$ in the following ways.

 \mathcal{I}_1 : The indicator (3.17) of the third order scheme is used to detect troubled cells. We expect that on smooth cells Ω_j one has $(S^3)_j^n = \mathcal{O}(h^3)$, whereas $(S^3)_j^n = \mathcal{O}(h^{-1})$ in presence of shocks. In particular, a cell Ω_j is marked for time-limiting if

$$(S^3)_j^n > \gamma_1,$$
 (3.18)

where γ_1 is a given threshold.

 \mathcal{I}_2 : The ratio of the indicators (3.16) and (3.17) is used to detect problematic cell Ω_j whenever

$$\frac{(S^3)_j^n}{(S^1)_j^n + \sigma} > \gamma_2, \tag{3.19}$$

where γ_2 is a given threshold and σ is a small quantity which prevents $(s^3)_j^n/(s^1)_j^n \sim \mathcal{O}(1)$ on constant regions of the solution. In fact, we expect that on a smooth cell Ω_j one has $(s^3)_j^n/(s^1)_j^n = \mathcal{O}(h^2)$, whereas $(s^3)_j^n/(s^1)_j^n = \mathcal{O}(1)$ if the solution is not regular on Ω_j .

 \mathcal{I}_3 : Both the detecting techniques \mathcal{I}_1 and \mathcal{I}_2 are used, so that a cell Ω_j is marked for time-limiting if

$$(S^3)_j^n > \gamma_1 \text{ and } \frac{(S^3)_j^n}{(S^1)_j^n + \sigma} > \gamma_2.$$
 (3.20)

We will compare these strategies in the numerical section. For now let us remark that \mathcal{I}_2 should be more robust than \mathcal{I}_1 since it involves ratios, but that it may fail on flat regions where both S^3 and S^1 can be floating point zeros. Thus in \mathcal{I}_3 we propose to combine the \mathcal{I}_1 and the \mathcal{I}_2 strategies. In fact, the \mathcal{I}_2 strategy may mark flat regions as irregular cells, i.e. where a shock appears, if $(S^3)_j^n = \mathcal{O}((S^1)_j^n)$. Then, the \mathcal{I}_1 strategy allows to detect the regularity of the solution.

The thresholds γ_1 and γ_2 are very important for the performance of the regularity detection. We will come back to this point in Section 4 when we discuss the values we choose for the two thresholds.

3.3.2 Time limited solution and the final Quinpi algorithm

Once the detector has been chosen, the flux-centered time-limiting is performed by computing the limited numerical flux $\mathcal{F}_{j+1/2}^{\mathsf{TL},(k)}$ for the face between Ω_j and Ω_{j+1} , for all stages $k=1,\cdots,s$, as

$$\mathcal{F}_{j+\frac{1}{2}}^{\mathsf{TL},(k)} = \begin{cases} \theta_k \mathcal{F}_{j+\frac{1}{2}}^{\star,(k)}, & \text{if either } \Omega_j \text{ or } \Omega_{j+1} \text{ is marked}, \\ \\ b_k \mathcal{F}_{j+\frac{1}{2}}^{(k)}, & \text{if both } \Omega_j \text{ and } \Omega_{j+1} \text{ are not marked}, \end{cases}$$
(3.21)



Figure 1: The algorithm of the Quinpi scheme for the computation of the solution in a single time-step. The DIRK routine, which provides the high order solution with the space limiters computed by the predictor, is followed by the time limiter which downgrades the numerical fluxes at the interface of an irregular cell.

where the acronym TL stands for Time-Limited. Notice that, since the predictor scheme is based on a composite backward Euler, the low order numerical fluxes $\mathcal{F}_{j\pm 1/2}^{\star,(k)}$ are available at each stage $k = 1, \dots, s$, for $j = 1, \dots, N$.

Thus, the high order numerical fluxes at the interfaces of a problematic cell Ω_j are replaced with the low order numerical fluxes reducing locally the order of the solution, which is then computed as

$$\overline{\mathbf{U}}_{j}^{n+1} = \overline{\mathbf{U}}_{j}^{n} - \frac{\Delta t}{h} \sum_{k=1}^{s} \left[\mathcal{F}_{j+\frac{1}{2}}^{\mathsf{TL},(k)} - \mathcal{F}_{j-\frac{1}{2}}^{\mathsf{TL},(k)} \right], \quad j = 1, \cdots, N.$$
(3.22)

The time-limiting procedure is repeated until the solution $\overline{\mathfrak{U}}^{n+1}$ is detected as smooth on each cell.

Overall, the update of the solution from time $n\Delta t$ to time $(n+1)\Delta t$ computed with the Quinpi scheme is summarized in Fig. 1.

4 Numerical simulations

All the schemes we consider in this section use the third order CWENOZ reconstruction without ghost cells, see [46], described in Definition 2.2 and 2.3 with parameters q = p = 2 in (2.3) and

$$\begin{cases} d_L = \frac{1}{8}, \ d_R = \frac{1}{8}, & \text{for the inner computational cells,} \\ \tilde{d} = \max\{h, 0.01\}, \ d = \frac{1}{4}, & \text{for the first/last computational cell} \end{cases}$$

For the purposes of this work, it is sufficient to consider the very simple Rusanov (local Lax-Friedrichs) numerical flux

$$\mathcal{F}(\mathbf{v},\mathbf{w}) = \frac{1}{2} (\mathbf{f}(\mathbf{v}) + \mathbf{f}(\mathbf{w}) - \alpha(\mathbf{w} - \mathbf{v})).$$
(4.1)

Here, α is the parameter of numerical viscosity. In explicit schemes one has to choose $\alpha = \max\{\|\mathbf{f}'(\mathbf{v})\|, \|\mathbf{f}'(\mathbf{w})\|\}$, where $\|\mathbf{f}'(\cdot)\|$ denotes the spectral radius of the Jacobian of the flux function **f**. In implicit schemes this choice would deserve more attention, especially if the fluid speed is much lower than the sound speed and if one is not interested in the acoustic waves. In this case one can use more appropriate speed estimates in the approximate Riemann solver as explained, e.g., in [21]. We will present our choice for each test problem. In any case, when assembling the Jacobian of **G** of (3.7) for the Newton step (3.8), we neglect the terms $\partial_{\mathbf{v}} \alpha$ and $\partial_{\mathbf{w}} \alpha$ in the derivative of \mathcal{F} and approximate it as

$$\partial_{\mathbf{v}} \mathcal{F}(\mathbf{v},\mathbf{w}) \approx \frac{1}{2} \mathbf{J}_{\mathbf{f}}(\mathbf{v}) + \frac{1}{2} \alpha \mathbb{I}_{m}$$
 and $\partial_{\mathbf{w}} \mathcal{F}(\mathbf{v},\mathbf{w}) \approx \frac{1}{2} \mathbf{J}_{\mathbf{f}}(\mathbf{w}) - \frac{1}{2} \alpha \mathbb{I}_{m}$

where J_f denotes the Jacobian of the exact flux function and \mathbb{I}_m is the $m \times m$ identity matrix. For simplicity and not to distract from the main focus of the paper, which is on the implicit time integration of high order schemes, we have used the simple Rusanov flux.

For the time integration, we employ the three stage third order DIRK scheme of [3] with Butcher tableau

λ	λ	0	0
$rac{(1+\lambda)}{2}$	$\frac{(1-\lambda)}{2}$	λ	0
1	$-\frac{3}{2}\lambda^2 + 4\lambda - \frac{1}{4}$	$\frac{3}{2}\lambda^2 - 5\lambda + \frac{5}{4}$	λ
	$-\frac{3}{2}\lambda^2 + 4\lambda - \frac{1}{4}$	$\frac{3}{2}\lambda^2 - 5\lambda + \frac{5}{4}$	λ

where $\lambda = 0.4358665215$. This method is A-stable and stiffly accurate, and therefore it is L-stable. As a consequence, the Butcher tableau (3.6) of the composite backward Euler

<u>Label</u>	Scheme description	Line type
Q3	Quinpi solution without time-limiting	—
$Q3_{\mathcal{I}1}$	Quinpi solution with \mathcal{I}_1 time-limiting strategy (3.18)	
$Q3_{\mathcal{I}2}$	Quinpi solution with \mathcal{I}_2 time-limiting strategy (3.19)	
$Q3_{\mathcal{I}3}$	Quinpi solution with \mathcal{I}_3 time-limiting strategy (3.20)	
CWZb3	Explicit CWENOZ scheme of [46]	-·-*·-·
Q3P1	Implicit scheme of [43] with cell-centered time-limiter	
Q3P1 _{MOOD}	Implicit scheme of [50] which uses MOOD as time-limiter	

Table 2: List of the schemes tested, labels and line type (for those that appear in the plots).

becomes

λ	λ	0	0
$rac{(1+\lambda)}{2}$	λ	$\frac{1-\lambda}{2}$	0
1	λ	$\frac{1-\lambda}{2}$	$\frac{1-\lambda}{2}$
	λ	$\frac{1-\lambda}{2}$	$\frac{1-\lambda}{2}$

Although this is not necessary for our construction, note that the sequence of the abscissae of this Runge-Kutta schemes is strictly increasing. Other choices of DIRK schemes are possible, e.g. see [34].

All numerical simulations are performed with the schemes described in Table 2. For the time-limited schemes, we choose the threshold parameters in (3.18) and (3.20) as $\gamma_1 = h$ and $\sigma = 10^{-10}$, respectively. Instead, γ_2 , cf. (3.19), will be presented for each numerical test and we will investigate the performance of the scheme based on its value. In general, the choices of the thresholds γ_1 and γ_2 are dictated by the decay of the numerical entropy indicators, cf. Section 3.3.1, and by the regions of the solution one aims to mark as irregular. Indeed, $\gamma_1 = h$ allows us to mark as irregular all regions of the solution where $S^3 > h$, i.e. those regions characterized by shocks, contact waves, and rarefaction corners. Similarly, a constant value of γ_2 allows us to distinguish smooth regions, where the indicator $S^3/S^1 \sim O(h^2)$, from irregular regions where $S^3/S^1 \sim O(1)$.

For the solution of the nonlinear systems in the predictor and in the corrector, we used the Newton-Raphson schemes (3.8) and (3.11), and a simple GMRES linear solver with a ILU0 preconditioner. For a detailed discussion on properties of iterative solvers in the context of implicit discretizations of hyperbolic equations we refer to [9].

4.1 Gas-dynamics problems

In this section, we test the third order Quinpi scheme on several problems based on the one-dimensional nonlinear Euler system for gas-dynamics

$$\partial_t \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix} + \partial_x \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix} = 0, \tag{4.2}$$

where ρ , v, p and E are the density, velocity, pressure and energy per unit volume of an ideal gas, whose equation of state is $E = \frac{p}{\gamma-1} + \frac{1}{2}\rho v^2$, where $\gamma = 1.4$. The eigenvalues of the Jacobian of the flux are $\lambda_1 = v - c$, $\lambda_2 = v$ and $\lambda_3 = v + c$, where $c = \sqrt{\gamma p/\rho}$ is the sound speed. For this system, we consider the entropy pair defined by the entropy function $\eta(\mathbf{u}) = -\rho \log\left(\frac{p}{(\gamma-1)\rho^{\gamma}}\right)$ and the entropy flux $\psi(\mathbf{u}) = -v\eta(\mathbf{u})$, see [30]. Since we aim to investigate the accuracy of Quinpi schemes on material waves, the parameter of numerical viscosity in (4.1) is chosen as $\alpha = \max\{|v^-|, |v^+|\}$, where v^- and v^+ denote the BED of the velocity of the gas.

4.1.1 Convergence test

We check the order of accuracy on the nonlinear system (4.2) by simulating the advection of a density perturbation. The initial condition is

$$(\rho, v, p) = (1 + 0.5\sin(2\pi x), 1, 10^{\kappa}), \tag{4.3}$$

so that the exact solution at a given time *t* is a traveling wave

$$(\rho, v, p) = (1 + 0.5\sin(2\pi(x - t)), 1, 10^{\kappa}), \qquad (4.4)$$

with $\kappa \in \mathbb{N}_0$. We run the problem up to the final time t = 1. We use periodic boundary conditions on the domain [0,1] and, thus, we test not only the scheme accuracy, but also the proposed treatment of reconstructions for cells at the boundary.

For this problem, the spectral radius is $\lambda_{\max} := \max_{i=1,2,3} |\lambda_i| = 1 + 10^{\kappa/2} \sqrt{2\gamma}$, where λ_i denotes the *i*-th eigenvalue of the system, which is preserved during the time evolution. For an explicit scheme the CFL stability condition would require to choose a grid ratio

$$\frac{\Delta}{h} \leq \frac{1}{\lambda_{\max}} \sim 10^{-\kappa/2}.$$

In order to test the accuracy of the implicit scheme, we run the simulation with the following grid ratio

$$\frac{\Delta t}{h} = 4,$$

and consider $\kappa = 0$ and $\kappa = 4$, corresponding to the Courant numbers $C = \lambda_{\max} \frac{\Delta t}{h} \approx 10.7$ and $C \approx 673.3$, respectively.

	$Q3_{\mathcal{I}1}$		Q3 ₁₂		Q3 ₁₃	
N	L^1 error	rate	L^1 error	rate	L^1 error	rate
40	1.28e - 2	_	0.26	_	1.28e - 2	_
80	1.74e - 3	2.88	0.19	0.48	1.74e - 3	2.88
160	2.21e - 4	2.97	0.12	0.69	2.21e - 4	2.97
320	2.78e - 5	2.99	6.44e - 2	0.84	2.78e - 5	2.99
640	3.48e - 6	3.00	3.40e - 2	0.92	3.48e - 6	3.00
1,280	4.35e - 7	3.00	1.73e - 2	0.97	4.35e - 7	3.00
2,560	5.44e - 8	3.00	8.77 <i>e</i> -3	0.98	5.44e - 8	3.00
	03-		03-		03	
	Q3 ₁₁		Q3 ₁₂		Q3 ₂	3
N	$Q3_{I1}$ L^{∞} error	rate	$Q3_{I2}$ L^{∞} error	rate	$Q3_{I}$ L^{∞} error	³ rate
N 40	$Q3_{I1}$ $L^{\infty} \text{ error}$ $2.02e-2$	rate	$Q3_{I2}$ L^{∞} error 0.42	rate	Q3 _{<i>I</i>} L^{∞} error 2.02 e -2	³ rate
N 40 80	$Q3_{I1}$ $L^{\infty} \text{ error}$ $2.02e-2$ $2.78e-3$	rate _ 2.86	$Q3_{I2}$ $L^{\infty} \text{ error}$ 0.42 0.29	rate - 0.49	Q3 $_{I}$ L ^{∞} error 2.02 e -2 2.78 e -3	³ rate - 2.86
N 40 80 160	Q 3_{I1} L ^{∞} error 2.02 <i>e</i> -2 2.78 <i>e</i> -3 3.55 <i>e</i> -4	rate - 2.86 2.97	Q 3_{12} L ^{∞} error 0.42 0.29 0.18	rate - 0.49 0.68	Q3 _{<i>I</i>} L^{∞} error 2.02 <i>e</i> -2 2.78 <i>e</i> -3 3.55 <i>e</i> -4	³ rate - 2.86 2.97
N 40 80 160 320	Q 3_{I1} L^{∞} error 2.02 e -2 2.78 e -3 3.55 e -4 4.46 e -5	rate - 2.86 2.97 2.99	Q 3_{12} L^{∞} error 0.42 0.29 0.18 0.11	rate 0.49 0.68 0.80	Q3 _{<i>I</i>} L^{∞} error 2.02 <i>e</i> -2 2.78 <i>e</i> -3 3.55 <i>e</i> -4 4.46 <i>e</i> -5	³ rate - 2.86 2.97 2.99
N 40 80 160 320 640	Q 3_{I1} L ^{∞} error 2.02e-2 2.78e-3 3.55e-4 4.46e-5 5.59e-6	rate - 2.86 2.97 2.99 3.00	Q 3_{12} L^{∞} error 0.42 0.29 0.18 0.11 5.71 e -2	rate - 0.49 0.68 0.80 0.89	Q3 _{<i>I</i>} L^{∞} error 2.02 <i>e</i> -2 2.78 <i>e</i> -3 3.55 <i>e</i> -4 4.46 <i>e</i> -5 5.59 <i>e</i> -6	³ rate - 2.86 2.97 2.99 3.00
N 40 80 160 320 640 1,280	Q 3_{I1} L^{∞} error 2.02 e -2 2.78 e -3 3.55 e -4 4.46 e -5 5.59 e -6 6.99 e -7	rate - 2.86 2.97 2.99 3.00 3.00	Q 3_{12} L^{∞} error 0.42 0.29 0.18 0.11 5.71 e -2 2.99 e -2	rate - 0.49 0.68 0.80 0.89 0.93	Q3 _{<i>I</i>} L^{∞} error 2.02 <i>e</i> -2 2.78 <i>e</i> -3 3.55 <i>e</i> -4 4.46 <i>e</i> -5 5.59 <i>e</i> -6 6.99 <i>e</i> -7	³ rate - 2.86 2.97 2.99 3.00 3.00

Table 3: Comparisons of the orders of convergence of Quinpi schemes on the initial condition (4.3) with $\kappa = 0$. The grid ratio is $\Delta t/h = 4$ which corresponds to a Courant number C = 10.7.

The L^1 - and L^{∞} -norm errors computed on the density component and the corresponding experimental orders of convergence are listed in Table 3 and Table 4 for the two different Courant numbers. The results are obtained with $\gamma_2 = 0.1$.

While the schemes $Q3_{I1}$ and $Q3_{I3}$ achieve the theoretical order of convergence, the scheme $Q3_{I2}$ exhibits, instead, first order accuracy. This behavior is explained by investigating the time-limiting procedure for $Q3_{I2}$. In fact, the marking strategy I_2 fails on flat regions where both S^3 and S^1 are floating point zeros. This causes the unnecessary limiting of all numerical fluxes, and thus the computed solution ends up coinciding with the first order predictor. Instead, the other detectors for time-limiting are able to signal the regularity of the solution, avoiding the unnecessary activation of the time-limiting. For these reasons, in the following we will not consider the scheme $Q3_{I2}$ anymore.

Furthermore, we observe that Table 3 and Table 4 present identical errors and rates of convergence for the methods $Q3_{I1}$ and $Q3_{I3}$. This happens because the time-limiting is indeed off on the smooth profile: the two strategies detect no irregular cells and coincide with the unlimited scheme Q3. Note that the magnitude of the errors is the same for $\kappa = 0$ and $\kappa = 4$, even though the latter uses a time-step which violets the explicit CFL by a factor of 60 with respect to the first one.

	$Q3_{I1}$		$Q3_{I2}$		Q3 _{I3}	
N	L^1 error	rate	L^1 error	rate	L^1 error	rate
40	1.21e-2	_	0.22	_	1.21e - 2	_
80	1.64e - 3	2.88	0.14	0.61	1.64e - 3	2.88
160	2.09e - 4	2.97	8.27e - 2	0.80	2.09e - 4	2.97
320	2.63e - 5	2.99	4.43e - 2	0.90	2.63e - 5	2.99
640	3.29e - 6	3.00	2.28e - 2	0.96	3.29e - 6	3.00
1,280	4.12e - 7	3.00	1.16e - 2	0.98	4.12e - 7	3.00
2,560	5.15e - 8	3.00	5.83e - 3	0.99	5.15e - 8	3.00
	$Q3_{\mathcal{I}1}$		$Q3_{I2}$		$Q3_{I3}$	
N	L^{∞} error	rate	L^{∞} error	rate	L^{∞} error	rate
40	1.89e - 2	_	0.35	_	1.89e - 2	_
80	2.58e - 3	2.87	0.23	0.61	2.58e - 3	2.87
160	3.29e - 4	2.97	0.13	0.81	3.29e - 4	2.97
320	4.13e - 5	2.99	6.91 <i>e</i> -2	0.90	4.13e - 5	2.99
640	5.17e - 6	3.00	3.58e - 2	0.95	5.17e - 6	3.00
1,280	6.47e - 7	3.00	1.82e - 2	0.98	6.47e - 7	3.00
2 560	8.09e - 8	3.00	9.17e - 3	0.99	8.09e - 8	3.00

Table 4: Comparisons of the orders of convergence of Quinpi schemes on the initial condition (4.3) with $\kappa = 4$. The grid ratio is $\Delta t/h = 4$ which corresponds to a Courant number C = 673.3.

4.1.2 Stiff Riemann problems

In the following, we test Quinpi schemes on stiff Riemann problems for the Euler system (4.2), namely for those problems characterized by $\frac{|v|}{|v|+c} \ll 1$, $\forall (x,t)$. In this case, we expect a strong separation between acoustic and material waves, because they travel with speeds having different magnitudes. In particular, the acoustic speeds would impose a strict constraint on the time-step

$$\frac{\Delta t_{\text{stab}}}{h} \leq \frac{1}{\max_{x} |v| + c}.$$

With implicit schemes, instead, the time-step is not dictated by stability and therefore we will choose it in order to reproduce accurately the slow material wave:

$$\frac{\Delta t_{\rm acc}}{h} = \frac{1}{|v_{\rm cw}|},$$

where v_{cw} denotes an estimate of the velocity of material waves. The ratio between the time-step Δt_{stab} required for stability and the time-step Δt_{acc} required for accuracy on the material wave, is a measure of the stiffness of the problem. We define the Courant number as

$$C = \frac{\Delta t_{\rm acc}}{\Delta t_{\rm stab}}.$$

We consider the following Riemann problems.



Figure 2: Exact density solution (thin gray solid line) and wave speeds (thick lines) of the Riemann problems at final time.

Test (a): Non-symmetric expansion problem.

$$(\rho, u, p) = \begin{cases} (1, -0.15, 1), & x < 0, \\ (0.5, 0.15, 1), & x \ge 0. \end{cases}$$
(4.5)

The exact density at time t = 1 is depicted in the left panel of Fig. 2 and is characterized by a 1-rarefaction moving with negative velocity, a 3-rarefaction moving with positive velocity, and a 2-contact wave with velocity $v_{cw} = -2.57 \times 10^{-2}$. For this problem, one has $\max_{(x,t)} |v| + c = 1.82$, thus an explicit scheme would require to choose a time-step Δt_{stab} such that

$$\frac{\Delta t_{\rm stab}}{h} \le 0.549.$$

With an implicit scheme, it is possible to overcome this stability restriction. In particular, choosing a time-step Δt_{acc} to meet the accuracy on the contact wave, one has

$$\frac{\Delta t_{\rm acc}}{h} = 6.66,$$

which implies a Courant number $C \approx 12.1$.

Test (b): colliding flow problem.

$$(\rho, v, p) = \begin{cases} (1.5, 0.5, 10), & x < 0, \\ (0.5, -0.5, 10), & x \ge 0. \end{cases}$$
(4.6)

The exact density at time t=1 is depicted in the center panel of Fig. 2 and is characterized by a 1-shock wave moving with negative velocity, a 3-shock wave moving with positive velocity, and a 2-contact wave with velocity $v_{cw} = 0.13$. For this problem, one has $\max_{(x,t)} |v| + c = 5.79$, thus an explicit scheme would require to choose

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a time-step Δt_{stab} such that

$$\frac{\Delta t_{\rm stab}}{h} \le 0.17.$$

With an implicit scheme, instead, one can choose

$$\frac{\Delta t_{\rm acc}}{h} = 2,$$

which implies a Courant number $C \approx 11.6$.

Test (c): modified Lax shock tube problem.

$$(\rho, v, p) = \begin{cases} (0.445, 0, 3.528), & x < 0, \\ (0.5, 0, 2.528), & x \ge 0. \end{cases}$$
(4.7)

Compared to the classical Lax shock tube problem, the initial condition is characterized by a zero left velocity and a higher right pressure. This allows a faster separation of the waves. The exact density at time t = 0.15 is depicted in the right panel of Fig. 2 and is characterized by a 1-rarefaction moving with negative velocity, a 3-shock moving with positive velocity, and a 2-contact wave with velocity $v_{cw} = 0.35$. For this problem, one has $\max_{(x,t)} |v| + c = 3.61$, thus an explicit scheme would require to choose a time-step Δt_{stab} such that

$$\frac{\Delta t_{\mathrm{stab}}}{h} \leq 0.28$$

With an implicit scheme, instead, one can choose

$$\frac{\Delta t_{\rm acc}}{h} = 2.83,$$

which implies a Courant number $C \approx 10.2$.

Here below we comment the results of all three tests. First, we investigate the role of the parameter threshold γ_2 , see (3.20), which defines the time-limiting strategy \mathcal{I}_3 . In particular, we study the accuracy of the resulting $Q3_{\mathcal{I}3}$ scheme on the contact wave for several values of γ_2 . In Fig. 3, we show the L^1 -errors as function of the number of cells. We observe that in all the proposed Riemann problems the choice $\gamma_2 = 1$ provides smaller errors. Therefore, all the numerical solutions computed with $Q3_{\mathcal{I}3}$ are considered with $\gamma_2 = 1$.

Next, we show the need for time-limiting. In Fig. 4 we compare the solutions computed without time-limiting (Q3) and with the two limiting strategies \mathcal{I}_1 and \mathcal{I}_3 . The figures clearly show that

time-limiting is needed to avoid spurious oscillations. In particular, the numerical solution of the colliding flows problem obtained with the Q3 scheme is not showed because it blows up at time t = 0.93;



Figure 3: L^1 -errors on the contact wave computed with the Q3₂₃ schemes on several grids and different values of the threshold parameter γ_2 .



(c) Still Eax at t = 0.15 with 10 = 000 certs.

Figure 4: Comparison between third order implicit schemes, without time-limiting and with different time-limiting strategies. The left-most panel shows the density solution. The other panels show zooms on the three waves. The Q3 solution of the colliding flows problem is not showed because it blows up before reaching the final time.



Figure 5: Structure of the solutions in the x-t diagram. Limited cell interfaces are marked at the time levels and locations where time-limiting is applied. The colors denote the loop number at which a cell interface is marked: interfaces limited at initial loops are marked in blue, while those limited at final loops are marked in red. These results can be compared with the data provided in Table 5.

Table 5: Statistics on the time-limiting procedure \mathcal{I}_3 of Quinpi schemes. These data are graphically depicted in Fig. 5.

(a) Non-symmetric expansion.				(b) Colliding flows	5.	(c) Stiff Lax.		
Q3 _{I3}				$Q3_{I3}$			$Q3_{I3}$	
Ν	max. number of limited fluxes	% limited time-steps	Ν	max. number of limited fluxes	% limited time-steps	Ν	max. number of limited fluxes	% limited time-steps
400 800 1,600	41 52 60	12.5 6.45 8.2	1,000 2,000 4,000	27 43 68	4 15 64.5	200 400 800	38 67 77	83.33 45.46 22.73

- strategy \mathcal{I}_1 , see (3.18), is more diffusive. Therefore, in the following we will consider the third strategy \mathcal{I}_3 only, see (3.20). Moreover, observe that the Q3_{*I*3} solution is accurate as the Q3 solution on the contact waves of the non-symmetric expansion and of the stiff Lax problems, while reducing the oscillations on the acoustic waves;
- the time-limiting is active on all the waves at initial times, i.e. when they separate themselves, cf. Fig. 5. At larger times, instead, limiting acts in particular on shock waves but stops when the waves are sufficiently separated. In Table 5 we observe that the percentage of limited steps is larger in problems where shock waves develop. Moreover, the number of limited fluxes is large because during the large time-step a wave propagates across more cells in a single time-step.

Finally, we compare the explicit scheme CWZb3 of [46], which has to be run with Δt_{stab} , and the implicit scheme Q3₇₃ run at Δt_{acc} , i.e. centered on the material wave. Fig. 6 shows that



(a) Non-symmetric expansion at t = 1 with N = 800 cells. The explicit solution is obtained with time-step 2.7×10^{-3} , whereas the implicit solution is computed with time-step 3.3×10^{-2} .



(b) Colliding flows at t=1 with N=2000 cells. The explicit solution is obtained with time-step 8.5×10^{-4} , whereas the implicit solution is computed with time-step 1.0×10^{-2} .



(c) Stiff Lax at t=0.15 with N=800 cells. The explicit solution is obtained with time-step 7.0×10^{-4} , whereas the implicit solution is computed with time-step 7.1×10^{-3} .

Figure 6: Comparison between third order implicit and explicit schemes. The left-most panel shows the density solution. The other panels show zooms on the three waves.

- as expected, the explicit scheme is more accurate than the implicit one on the acoustic waves;
- on the contact wave, which is the one we are focusing on, the implicit scheme is more accurate than the explicit one and runs with a time-step that is about 10 times larger.

4.2 Low-Mach problems

In this section, we test the performance of Quinpi schemes on low-Mach problems, namely such that

$$Ma:=\frac{|v|}{c}\ll 1, \quad \forall (x,t).$$

Here, the aim is to show that the implicit framework we propose has the ability to guarantee stability and accuracy on such problems. We do not claim that the Quinpi approach is asymptotic preserving, e.g. see [23].

In order to describe the low-Mach number limit, we consider the rescaled compressible Euler system

$$\partial_t \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix} + \partial_x \begin{pmatrix} \rho v \\ \rho v^2 + \frac{1}{\varepsilon^2} p \\ v(E+p) \end{pmatrix} = 0, \tag{4.8}$$

with the equation of state $E = \frac{p}{\gamma-1} + \frac{\varepsilon^2}{2}\rho v^2$, $\gamma = 1.4$. The parameter ε describes the Mach number within the non-dimensionalized system, and one has $\operatorname{Ma} = \varepsilon/\sqrt{\gamma}$. System (4.8) is hyperbolic with eigenvalues $\lambda_1 = v - c/\varepsilon$, $\lambda_2 = v$, $\lambda_3 = v + c/\varepsilon$. For a more detailed discussion on the low-Mach number scaling we refer, e.g., to [11,36]. Although in the following we consider only smooth problems, we still choose the parameter of numerical viscosity in (4.1) according to the material speed, namely $\alpha = \max\{|v^-|, |v^+|\}$, where v^- and v^+ denote the BED of the velocity of the gas.

4.2.1 Convergence test

We compute the experimental order of convergence of the Quinpi scheme on the computational domain [-2.5,2.5] using the smooth initial condition [11]

$$\rho(x,0) = \left(1 + \varepsilon \frac{(\gamma - 1)u(x,0)}{2\sqrt{\gamma}}\right)^{\frac{2}{\gamma - 1}}, \quad u(x,0) = \sin\left(\frac{2\pi x}{5}\right), \quad p(x,0) = \rho(x,0)^{\gamma}.$$

The order of accuracy of the scheme is investigated at final time t=0.3 for $\varepsilon_1=0.8$, $\varepsilon_2=0.3$, and at final time t=0.01 for $\varepsilon_3=10^{-4}$. For this problem, $|v| \le 1$, $\forall (x,t)$, and $\lambda_{\max}(\varepsilon) := \max_{(x,t)} |v| + c/\varepsilon$ is

$$\lambda_{\max}(0.8) = 2.6786, \quad \lambda_{\max}(0.3) = 5.1439, \quad \lambda_{\max}(10^{-4}) = 1.1833 \times 10^{4}$$

For an explicit scheme the CFL stability condition would require to choose a grid ratio

$$\begin{aligned} \frac{\Delta t_{\varepsilon_1}}{h} &\leq \frac{1}{\lambda_{\max}(0.8)} \approx 0.3733, \\ \frac{\Delta t_{\varepsilon_2}}{h} &\leq \frac{1}{\lambda_{\max}(0.3)} \approx 0.1944, \\ \frac{\Delta t_{\varepsilon_3}}{h} &\leq \frac{1}{\lambda_{\max}(10^{-4})} \approx 8.45 \times 10^{-5} \end{aligned}$$

	$\varepsilon = 0.8$		$\varepsilon = 0.3$		$\varepsilon = 10^{-4}$	
Ν	L^1 error	rate	L^1 error	rate	L^1 error	rate
160	5.58e - 2	0.98	2.40e - 2	1.75	2.24e - 4	0.26
320	1.23e - 2	2.18	5.61e - 3	2.10	5.41e - 5	2.05
640	2.26e - 3	2.45	8.45e - 4	2.73	7.64e - 6	2.82
1,280	2.95e - 4	2.94	1.11e - 4	2.93	9.61e - 7	2.99
	$\varepsilon = 0.8$	8	$\varepsilon = 0.3$		$\varepsilon \!=\! 10^{-4}$	
N	L^{∞} error	rate	L^{∞} error	rate	L^{∞} error	rate
160	6.34e - 2	0.77	1.85e - 2	1.45	7.05e - 5	0.26
320	1.59e - 2	1.99	5.80e - 3	1.67	1.72e - 5	2.04
640	3.08e - 3	2.37	1.01e - 3	2.52	2.42e - 6	2.83
1,280	4.04e - 4	2.93	$1.39e{-4}$	2.87	3.04e - 7	2.99

Table 6: Experimental order of convergence of Quinpi scheme on the low-Mach problem with Courant number C = 20.

We overcome the CFL condition running the simulation using the following grid ratios:

$$\frac{\Delta t_{\varepsilon_i}}{h} = \frac{C}{\lambda_{\max}(\varepsilon_i)}, \quad i = 1, 2, 3,$$

with Courant number C = 20.

Density errors for the different values of the Mach number are listed in Table 6 and computed with the Quinpi scheme Q3. The theoretical third order of convergence is observed.

4.2.2 Two colliding acoustic pulses

We consider the two colliding acoustic pulses taken from [36]. The initial condition is given by

$$\rho(x,0) = \rho_0 + \frac{\varepsilon \rho_1}{2} \left(1 - \cos\left(\frac{2\pi x}{L}\right) \right), \quad \rho_0 = 0.955, \quad \rho_1 = 2,$$

$$u(x,0) = -\frac{u_0}{2} \operatorname{sign}(x) \left(1 - \cos\left(\frac{2\pi x}{L}\right) \right), \quad u_0 = 2\sqrt{\gamma},$$

$$p(x,0) = p_0 + \frac{\varepsilon p_1}{2} \left(1 - \cos\left(\frac{2\pi x}{L}\right) \right), \quad p_0 = 1, \quad p_1 = 2\gamma,$$

on the computational domain [-L, L], $L = 2/\varepsilon$. We use periodic boundary conditions and, thus, we also test the proposed treatment of [46] which does not require ghost cells.

We choose $\varepsilon_1 = 1/11$ and $\varepsilon_2 = 10^{-4}$. In both cases, $\max_{(x,t)} |v| = 2.3656$, whereas $\lambda_{\max}(\varepsilon_1) = 16.038$ and $\lambda_{\max}(\varepsilon_2) = 1.21 \times 10^4$. Therefore, using an explicit scheme would

require to choose a time-step

$$\frac{\Delta t_{\varepsilon_1}}{h} \le 6.24 \times 10^{-2}, \quad \frac{\Delta t_{\varepsilon_2}}{h} \le 8.26 \times 10^{-5}.$$

With an implicit scheme, instead, we can choose

$$\frac{\Delta t_{\varepsilon_1}}{h} = \frac{1}{\max_{(x,t)} |v|} = 4.23 \times 10^{-1},$$

which corresponds to a Courant number C = 6.78. We employ the same Courant number for ε_2 which determines

$$\frac{\Delta t_{\varepsilon_2}}{h} = 5.59 \times 10^{-4}.$$

In Fig. 7 we show the pressure profile obtained using the third order Quinpi scheme Q3 without time-limiting at final times t = 0.815 and t = 1.63, with N = 440 cells. The solution is compared with the third order explicit scheme CWZb3.

4.3 Scalar problems

Finally, we provide some numerical simulations on scalar conservation laws. In fact, although this work is mostly focused on hyperbolic systems, the Quinpi framework introduced in [43], and further extended in [50], was based on very different time-limiting procedures. Therefore, this section aims to investigate the performance of the proposed MOOD limiter combined with the numerical entropy production indicator on standard scalar problems. In all the scalar problems, the numerical viscosity in (4.1) is chosen as $\alpha = \max\{|f'(v)|, |f'(w)|\}$ where *f* is the flux function. All simulations are performed with $\gamma_2 = 0.1$.

4.3.1 Linear transport

We consider the linear scalar conservation law

$$\partial_t u(x,t) + \partial_x u(x,t) = 0, \tag{4.9}$$

for $(x,t) \in [-1,1] \times (0,2]$, with periodic boundary conditions in space and discontinuous initial conditions:

$$u_0(x) = \sin(\pi x) + \begin{cases} 3, & -0.4 \le x \le 0.4, \\ 0, & \text{otherwise,} \end{cases}$$
(4.10a)

$$u_0(x) = \begin{cases} 1, & -0.25 \le x \le 0.25, \\ 0, & \text{otherwise.} \end{cases}$$
(4.10b)

Here, we consider the entropy function $\eta(u) = \frac{u^2}{2}$ and the entropy flux $\psi(u) = \frac{u^2}{2}$.



(a) $\varepsilon = \frac{1}{11}$, t = 0.815 (left) and t = 1.63 (right). The explicit solution is obtained with time-step 0.624, whereas the implicit solution is computed with time-step 4.23.



(b) $\varepsilon = 10^{-4}$, t = 0.815 (left) and t = 1.63 (right). The explicit solution is obtained with time-step 7.5×10^{-4} , whereas the implicit solution is computed with time-step 5.1×10^{-3} .

Figure 7: Two colliding pulses problem with n = 440 cells. The solutions show the initial pressure and the final pressure profiles obtained with the explicit CWZb3 and the implicit Q3 third order schemes.

The results are reported in Fig. 8, computed with two different Courant numbers, i.e. $\Delta t/h = 5$ and $\Delta t/h = 10$ on 400 space cells. As introduced in Table 2, the solution labeled by Q3P1 refers to the scheme of [43], whereas the solution labeled by Q3P1_{MOOD} refers to the one of [50]. We note that the scheme Q3_{I3} performs better than the other two implicit schemes on the discontinuous sinusoidal profile since it produces lower dissipation around the jump discontinuities. However, the low-dissipation properties leads to lightly larger oscillations around the discontinuities of the double-step profile. In par-



Figure 8: Linear transport equation (4.9) with initial condition (4.10a) (left panels) and initial condition (4.10b) (right panels). All the solution are obtained on a grid of 400 cells with two Courant numbers. The label Q3P1 refers to the scheme of [43] with mass redistribution.

ticular, with $\Delta t/h = 10$, the upper flat part of the solution is not reproduced by all the schemes.

4.3.2 Burgers' equation

Next, we compare the schemes on the nonlinear Burgers' equation

$$\partial_t u(x,t) + \partial_x \left(\frac{u^2(x,t)}{2}\right) = 0,$$
(4.11)

for $x \in [-1,1]$, with periodic boundary conditions in space. Here, we consider the entropy function $\eta(u) = \frac{u^2}{2}$ and the entropy flux $\psi(u) = \frac{u^3}{3}$.



Figure 9: Burgers' equation (4.11) with initial condition (4.12) on 400 cells with $\Delta t/h = 3$ (top panels) and $\Delta t/h = 10$ (bottom panels), at three different times.

We test (4.11) on the smooth initial condition

$$u_0(x) = 0.2 - \sin(\pi x) + \sin(2\pi x), \tag{4.12}$$

computing the solution at three different times, namely $t = 1/2\pi$, 0.6, 1, and Courant numbers $\Delta t/h = 3,10$, on a space grid of 400 cells. The exact solution is characterized by the formation of two shocks which collide developing a single discontinuity.

The numerical results are depicted in Fig. 9. We observe that all schemes do not produce spurious oscillations at time $t = 1/(2\pi)$, just before the two shocks appear, but the scheme Q3P1 is more diffusive. At t = 0.6, Q3P1_{MOOD} exhibits a small undershoot on the left shock, clearly visible in the zoom box, which is not present in the solution computed by Q3₂₃. Increasing the Courant number produces more dissipation within the numerical schemes Q3P1 and Q3P1_{MOOD}, whereas Q3₂₃ is more accurate and does not exhibit spurious oscillations.

5 Conclusions and future research

In this work, we have proposed a framework for the high order implicit integration of hyperbolic systems of conservation laws. The approach is based on the recent work [43],

which was focused on scalar conservation laws. The crucial idea of Quinpi is to use a first order implicit predictor to freeze the nonlinear space limiters in the CWENO third order reconstruction, because such a predictor gives a pre-estimate of the WENO weights, which will be used by the higher order method. Space limiting however is not enough, and it is necessary to perform limiting in time, based on mixing the computed high order solution and the low order predictor. As a consequence, the nonlinearity of the scheme is linked only to the nonlinearity of the flux, while space and time-limiting need not enter into the nonlinear iterations needed to update the solution.

At variance with [43], we have designed a flux-centered conservative time-limiting procedure inspired by the MOOD technique [13, 14], where the numerical entropy residual [40,41] has been used as indicator of non-smooth solutions and troubled cells. This is more robust than the previous cell-centered limiter that required a conservative correction after the time-limiting.

We mention that this implicit approach has not been tailored to the solution of a specific problem. We expect the new scheme to have impact in many applications described by stiff problems, in particular in those cases in which the phenomena of interest travel much slower than the fastest wave speeds. Our tests show that implicit schemes can provide a better accuracy compared to explicit ones on material waves, even running with a time-step about 10 times larger.

Future work will involve the study of the Quinpi approach combined with different time integration methods, for instance using hybrid BDF schemes, extensions to higher order, and possibly investigation of analytical properties of the scheme. Further attention will also be paid to improving the efficiency of the scheme. To this end, we will investigate ad-hoc linear solvers and preconditioners, especially in view of higher dimensional applications, and we plan to improve the time-limiting procedure. We expect to make the computational cost scale more favorably for the implicit approach.

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