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A Discontinuous Galerkin Method for Pricing American Options Under the Constant Elasticity of Variance Model

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Abstract. The pricing of option contracts is one of the classical problems in Mathematical Finance. While useful exact solution formulas exist for simple contracts, typically numerical simulations are mandated due to the fact that standard features, such as early-exercise, preclude the existence of such solutions. In this paper we consider derivatives which generalize the classical Black-Scholes setting by not only admitting the early-exercise feature, but also considering assets which evolve by the Constant Elasticity of Variance (CEV) process (which includes the Geometric Brownian Motion of Black-Scholes as a special case). In this paper we investigate a Discontinuous Galerkin method for valuing European and American options on assets evolving under the CEV process which has a number of advantages over existing approaches including adaptability, accuracy, and ease of parallelization.

AMS subject classifications: 91G20, 91G60, 65M70

Key words: Option pricing, PDE methods, CEV process, Discontinuous Galerkin method.

1 Introduction

The pricing of option contracts is one of the classical problems in Mathematical Finance [11, 14, 20]. While useful exact solution formulas exist for simple contracts (e.g., vanilla European calls/puts) [11, 14, 20], typically numerical simulations are mandated due to the fact that standard features, such as early-exercise (e.g., an American contract), preclude the existence of such solutions. We point out the work of Zhu [23] which explicitly describes a solution procedure for American options, however, the method is both quite

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complicated and very slowly converging. In this paper we consider derivatives which generalize the classical Black-Scholes setting by not only admitting the American feature, but also considering assets which evolve by the Constant Elasticity of Variance (CEV) process (which includes the Geometric Brownian Motion-GBM-of Black-Scholes as a special case) [2,6,12,22]. We consider American options not only for their intrinsic interest, but also because they are more common than Europeans [11]. We consider the CEV model as this is one of the most popular methods for capturing the "volatility smile" implied by option prices in the financial markets [22].

While there are myriad techniques for numerically simulating such options, they typically fall into one of three categories: Binomial methods, Monte Carlo methods, and PDE methods [11,14,20]. Binomial methods are both quite flexible (American features can be accommodated nearly seamlessly) and simple to implement, but have a rather slow (errors like $1/M$ in M, the number of nodes [11]) and non-monotonic convergence. The interested reader is referred to the illustrative computations in Chapter 16 of [11] for representative simulations, and for a concrete implementation (which we utilize later in this paper) we refer the reader to Lu & Hsu [17]. The Monte Carlo method is also extremely flexible (particularly for contracts with complex features or those based upon multiple assets) and relatively straightforward to code. These approaches are more difficult to extend to American contracts and suffer from *extremely* slow rates of convergence (errors proportional to $1/\sqrt{M}$ for M samples [11]) which necessitates sophisticated "variance reduction" techniques [11,14] to generate competitive algorithms. We point out that Glasserman [10] provides a Monte Carlo method for the CEV process. Additionally, we refer the reader to the work of Hsu, Lin, & Lee [12], Lo, Tang, Ku, & Hui [18], and Knessl & Xu [16] for further results on pricing options on assets evolving under the CEV process.

PDE methods have their own list of advantages and shortcomings which render them the method of choice for pricing options under certain sets of conditions [11,14,20]. PDE methods solve the Black-Scholes PDE (or its generalization) *directly* which leads to the value of the option for *all* possible asset prices and times between inception and expiry of the contract. While this certainly is more information than is typically required, it does mean that sensitivity information from the Greeks can approximated with little or no extra cost. Additionally, as PDE methods admit the possibility of *high-order* simulation for Europeans (i.e., featuring errors which decay like C/M^p, integer $p\geq 1$, or even $Ce^{-\kappa M}$, $\kappa>0$, for M unknowns), the number of degrees of freedom required to deliver an option price with a fixed tolerance may be comparable to (or smaller than) that required by binomial or Monte Carlo simulation.

In this paper we investigate a PDE method for valuing both European and American options on assets evolving under the CEV process which has a number of advantages over existing approaches including adaptability, arbitrary-order accuracy (for Europeans), and ease of parallelization. Approaches to valuing options based upon Finite Difference (FDMs) and Finite Element Methods (FEMs) are, by now, classical [11,14,20] and typically couple either second-order central differencing to implicit time stepping (resulting in the Crank-Nicolson method), or piecewise-linear basis functions to implicit time