Commun. Comput. Phys.
doi: 10.4208/cicp.110214.101014a

Vol. **17**, No. 3, pp. 779-807
March 2015

Efficient and Accurate Numerical Solutions for Helmholtz Equation in Polar and Spherical Coordinates

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Received 11 February 2014; Accepted (in revised version) 10 October 2014

Abstract. This paper presents new finite difference schemes for solving the Helmholtz equation in the polar and spherical coordinates. The most important result presented in this study is that the developed difference schemes are pollution free, and their convergence orders are independent of the wave number k. Let h denote the step size, it will be demonstrated that when solving the Helmholtz equation at large wave numbers and considering kh is fixed, the errors of the proposed new schemes decrease as h decreases even when k is increasing and $kh>1$.

AMS subject classifications: 65N06, 65N15, 65N22

Key words: Helmholtz equation, high wave number, polar coordinate, spherical coordinate, pollution free, high order difference scheme.

1 Introduction

In this paper, we study the Helmholtz equation in the exterior domain:

$$-\Delta u - k^2 u = 0, \quad \text{in } \mathbb{R}^d \backslash D, \tag{1.1}$$

$$u|_{\partial D} = g_1, \tag{1.2}$$

$$\partial_r u - jku = o\left(||x||^{\frac{1-d}{2}}\right), \text{ as } ||x|| \to \infty, \tag{1.3}$$

where D is a bounded domain in $\mathbb{R}^d$ $(d=2,3)$, k is the wave number, ∂_r denotes the radial derivative, g_1 is a given function and $j^2=-1$.

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The problem (1.1)-(1.3) appears in many applications, such as the electromagnetic wave scattering, acoustic and geophysical problems. Developing efficient and accurate numerical schemes to solve the Helmholtz equation at large wave number k is a very challenging task, and it has attracted a great of attention for a long time. Not only the problem has an unbounded domain, but the solution also becomes highly oscillatory for cases with large k. Many literatures have been devoted to the Helmholtz equation, see [3,7,8,13,14,18,26,29,31–35] for the finite difference methods, [1,2,4–6,16,21,24,25,30,38] for the finite element methods, [27,28] for the spectral method, [19,20] for the boundary element method and [15,17] for other techniques. An essential step to solve the above problem is to consider a bounded computational domain (see [3,19,27,28,32,35]):

$$-\Delta u - k^2 u = 0, \quad \text{in } \Omega := B \backslash D, \tag{1.4}$$

$$u|_{\partial D} = g_1, \tag{1.5}$$

$$\left(\partial_r u - jku\right)|_{\partial B} = g_2, \tag{1.6}$$

where $B \in \mathbb{R}^d$ $(d = 2,3)$ is a sufficiently large ball containing D and g_2 is a given function.

When approximating the Helmholtz equation (1.4)-(1.6) numerically, it will result a numerical wave number which will disperse in non-dispersive media for most of the existing numerical schemes. The numerical dispersion directly relates to the pollution error of the computed solution, which is known as a "pollution effect" (see [5,21]). The "pollution effect" causes a serious problem when the wave number k is very large (see [1,21,24,25,34]). It should be noted that while a numerical scheme works well for solving the problem (1.4)-(1.6) for small k, the accuracy and performance usually deteriorate significantly as k increases. Due to the "pollution effect", the computed results likely become totally unacceptable when k is very large. To overcome this problem, many investigations have been reported in the past decades. For the finite difference methods, Chen et al. [32,35] proposed methods to minimize the numerical dispersion, and the use of higher order finite difference schemes were also studied in [18,29,31]. For the finite element methods, Babuška et al. [5] designed a generalized finite element method to ensure that the pollution effect is minimal, further development can be found in [2,30]. Assuming that kh is fixed, Ihlenburg and Babuška [24,25] proposed the h-p finite element method to solve the equations (1.4)-(1.6), in which the "pollution effect" can be reduced as p increases or h decreases. This approach was continued and reported in [1,38]. In [27,28], Shen and Wang considered the problem in polar and spherical coordinates, they analyzed the spectral method and derived a complete error for the Helmholtz equation in 2D and 3D. Other techniques using the discontinuous Galerkin method and the discrete singular convolution algorithm had also been investigated for problems with large wave numbers (see [15,17]).

Generally speaking, in order to ensure the bound of the relative error for the numerical solution of (1.4)-(1.6), it is necessary to impose the condition $k^\beta (kh)^\gamma = constant$ where $\beta > 0$, $\gamma \geq 0$ are real numbers and h is the mesh size. For example, $\beta = 2$ and $\gamma = 2$ for the standard finite difference method, $\beta = 2$ and $\gamma = 4$ for the fourth order compact finite