

# Convergence Study of Moment Approximations for Boundary Value Problems of the Boltzmann-BGK Equation

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**Abstract.** The accuracy of moment equations as approximations of kinetic gas theory is studied for four different boundary value problems. The kinetic setting is given by the BGK equation linearized around a globally constant Maxwellian using one space dimension and a three-dimensional velocity space. The boundary value problems include Couette and Poiseuille flow as well as heat conduction between walls and heat conduction based on a locally varying heating source. The polynomial expansion of the distribution function allows for different moment theories of which two popular families are investigated in detail. Furthermore, optimal approximations for a given number of variables are studied empirically. The paper focuses on approximations with relatively low number of variables which allows to draw conclusions in particular about specific moment theories like the regularized 13-moment equations.

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**Key words:** Kinetic gas theory, Hermite expansion, channel flow.

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## 1 Introduction

The Boltzmann equation [4] describes the behavior of rarefied gases [3] on the basis of a probability density function defined on the phase space of particles including physical space and particle velocity. Moment methods have been originally explored by Harold Grad, see [9], to solve the Boltzmann equation in an approximate way. In moment approximations a polynomial expansion of the distribution function is used and the Boltzmann equation is projected onto polynomial basis functions of the velocity space yielding partial differential equations for the coefficients of the expansion.

Over the last decades moment theories have been investigated and further developed in various directions. Within the frame work of extended thermodynamics the

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text book [19] demonstrates how moment approximations can be used to describe gas processes. Special mathematical properties of moment systems have been pointed out, e.g., in [15]. Moment equations come with the advantage that relatively low order approximations already give interesting results and there exist various particular moment equations which have been studied in detail, for example, Grad's original 13-moment case [8] and Levermore's 10-moment-closure [16]. A regularization technique that overcomes certain artefacts in moment closures have been suggested in [25] resulting in the regularized 13-moment-equations (R13), see also the text book [23]. Regularized moment equations have been subsequently derived based on the 10-moment-case in [17], the 26-moment-equations in [12] and used for high order approximation in a numerical context [2]. Attempts to develop low order moment equations that obey a physical entropy law or H-theorem are found in [10, 18, 21, 22, 26]. Modifications of the moment method for gases can also be found in [6, 14] with further developments in e.g., [20]. Moment approximations are also used in other kinetic contexts, see e.g., [7], for general kinetic equations, and [5] for applications to radiation therapy.

The central question when using moment approximations is: how many moments are needed to simulate a certain process with sufficient accuracy? Since there is no general mathematical theory which can be used to answer this question, empirical studies like in [1] are typically conducted to gain understanding into the convergence behavior of moment approximations for specific processes. However, in the context of boundary value problems no such study exists, and it remains entirely unclear how moment equations cope with exponential Knudsen layers at the wall. Various investigations on channel flows with low order moment systems can be found, e.g., in [11, 13, 28–30, 32, 34] but a detailed convergence study of moment systems for gases with increasing number of variables is missing. A model system with large number of variables is considered in [24].

This paper will consider large moment approximations applied to the linear BGK equation for two shear flow and two heat conduction scenarios in one space dimension. The focus will be on low order approximations and the result will allow to re-examine the capability and accuracy of existing moment theories.

The paper is organized as follows: The linear kinetic setting of four different boundary value problems is presented in Section 2, while the generic moment approximation approach for the linear setting is detailed in Section 3. The moment equations are specialized to the boundary value problems in Section 4 and the field solution of some exemplary cases are presented. Section 5 presents and discusses the empirical error analysis of the moment approximations of the four channel processes. The paper ends with a conclusion.

## 2 Kinetic problem statement

### 2.1 Equations

We will consider the BGK equation