Commun. Comput. Phys.
doi: 10.4208/cicp.290914.030615a

Vol. **19**, No. 3, pp. 603-631
March 2016

Development of a Combined Compact Difference Scheme to Simulate Soliton Collision in a Shallow Water Equation

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Communicated by Wei Shyy

Received 29 September 2014; Accepted (in revised version) 3 June 2015

Abstract. In this paper a three-step scheme is applied to solve the Camassa-Holm (CH) shallow water equation. The differential order of the CH equation has been reduced in order to facilitate development of numerical scheme in a comparatively smaller grid stencil. Here a three-point seventh-order spatially accurate upwinding combined compact difference (CCD) scheme is proposed to approximate the first-order derivative term. We conduct modified equation analysis on the CCD scheme and eliminate the leading discretization error terms for accurately predicting unidirectional wave propagation. The Fourier analysis is carried out as well on the proposed numerical scheme to minimize the dispersive error. For preserving Hamiltonians in Camassa-Holm equation, a symplecticity conserving time integrator has been employed. The other main emphasis of the present study is the use of $u-P-\alpha$ formulation to get non-dissipative CH solution for peakon-antipeakon and soliton-anticuspon head-on wave collision problems.

AMS subject classifications: 35L05, 35J05, 65M06

Key words: Camassa-Holm equation, seventh-order spatially accurate, CCD, symplecticity, peakon-antipeakon, soliton-anticuspon.

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1 Introduction

Many weakly nonlinear dispersive partial differential equations have been known to permit soliton solutions [1]. Complex interaction amongst solitons in these physical systems has attracted considerable attention in the past. Soliton discovered experimentally by Russel is defined as a single humped wave of bulge water. Take the Korteweg-de Vries (KdV) equation $u_t+6uu_x+u_{xxx}=0$ as an example, solitary wave is formed due to the balance between the weak nonlinear steepening effect of uu_x and the linear dispersion effect of u_{xxx}. The nonlinear term in this equation tends to steepen the solution but the dispersion term is prone to spread it out at the same time. These localized and highly stable solitons amenable to KdV equation can retain its identity upon interaction. Because of the ability of preserving wave shape and speed, the collision of KdV solutions is classified to be of an elastic type.

Besides the celebrated KdV equation, the completely integrable Camassa-Holm (CH) equation [2] has also received considerable attention in the past two decades. Provided that an initial data is defined in the Sobolev space $H^s(\Omega)$ for $s>\frac{3}{2}$, CH equation is locally well-posed [3]. The reason for investigating this Cauchy problem is rooted in its possession of a rich geometric solution structure. Camassa-Holm equation investigated under a permanent wave motion has a global strong solution. In addition, this equation permitting blow-up solution can be used to model wave breaking [3]. For an initial data of the $H^1(\mathcal{R})$ type, Camassa-Holm equation is also amenable to global weak solution [4].

A smooth solution of CH equation can be compressed to form a jump in the solution in finite time [2] due to the occurrence of nonlinear terms in CH equation. In the presence of peakon solution, the CH solution computed at $\kappa=0$ exhibits a discontinuous first derivative at the crest. Any numerically introduced high-frequency dispersion error near these peaks can considerably deteriorate simulation quality [5]. It is natural to apply a compact scheme [6] to get a better understanding of the nonlinear and dispersive natures. In addition, the Weighted Essentially Non-Oscillatory (WENO) scheme [7] is also desired to avoid the oscillatory solutions.

Application of the finite difference scheme in [8] can properly model the peakon-antipeakon interaction. The pseudospectral scheme developed by Kalisch and Lenells [9] has been shown to be effective in predicting the solution of CH equation. Peakon solutions have been predicted more correctly by Artebrant and Schroll [10] using the adaptive upwinding finite volume discretization method. One can also apply the local discontinuous Galerkin method to predict the CH solution [5]. In addition to the above-mentioned methods, multi-symplectic method [11], energy-conserving Galerkin method [12], Hamiltonian-conserving Galerkin method [13] and self-adaptive mesh method [8] have been also developed to solve the CH equation with great success.

The rest of this paper is organized as follows. Section 2 describes the nonlinear CH equation and its remarkable mathematical features. This equation containing the third-order derivative term is then transformed to its equivalent nonlinear system of equations. This equivalent differential system of equations consists of one equation with the reduced