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Runge-Kutta Discontinuous Galerkin Method with a Simple and Compact Hermite WENO Limiter on Unstructured Meshes

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Abstract. In this paper we generalize a new type of compact Hermite weighted essentially non-oscillatory (HWENO) limiter for the Runge-Kutta discontinuous Galerkin (RKDG) method, which was recently developed in [38] for structured meshes, to two dimensional unstructured meshes. The main idea of this HWENO limiter is to reconstruct the new polynomial by the usage of the entire polynomials of the DG solution from the target cell and its neighboring cells in a least squares fashion [11] while maintaining the conservative property, then use the classical WENO methodology to form a convex combination of these reconstructed polynomials based on the smoothness indicators and associated nonlinear weights. The main advantage of this new HWENO limiter is the robustness for very strong shocks and simplicity in implementation especially for the unstructured meshes considered in this paper, since only information from the target cell and its immediate neighbors is needed. Numerical results for both scalar and system equations are provided to test and verify the good performance of this new limiter.

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Key words: Runge-Kutta discontinuous Galerkin method, HWENO limiter, unstructured mesh.

1 Introduction

In this paper we consider solving the two dimensional conservation law, given by

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$$\begin{cases} u_t+f(u)_x+g(u)_y=0, \\ u(x,y,0)=u_0(x,y), \end{cases} \tag{1.1}$$

using the Runge-Kutta discontinuous Galerkin (RKDG) method [6–9] on unstructured triangular meshes. RKDG methods use explicit, nonlinearly stable high order Runge-Kutta methods [33] to discretize the temporal variable and the DG methods to discretize the spatial variables, with exact or approximate Riemann solvers as interface fluxes. For a detailed discussion on DG methods for solving conservation laws, we refer the readers to the review paper [10] and the books and lecture notes [5, 15, 21, 32].

DG methods can compute the numerical solution to (1.1) without further modification provided the solution either is smooth or contains weak discontinuities. However, for problems containing strong shocks or contact discontinuities, there are spurious oscillations in the numerical solution near these discontinuities, which may cause nonlinear instability. One common strategy to control these oscillations is to apply nonlinear limiters to RKDG methods. Many limiters have been studied in the literature for RKDG methods, such as the *minmod* type total variation bounded (TVB) limiter [6–9], the moment based limiter [3] and an improved moment limiter [4] and so on. These limiters belong to the slope type limiters and they do control oscillations very well at the price of possibly degrading the accuracy of the numerical solution at smooth extrema. Another type of limiters is the WENO type limiters, which are based on the weighted essentially non-oscillatory (WENO) methodology [14,16,17,23] and can achieve both high-order accuracy and non-oscillatory property near discontinuities. This type of limiters includes the WENO limiter [27, 36] and the HWENO limiter [24, 26, 29], which use the classical WENO finite volume methodology for reconstruction and thus require a wide stencil, especially for higher order methods. Therefore, it is difficult to implement these limiters for multi-dimensional problems, especially on unstructured meshes. Moreover, these limiters may have the issue of negative linear weights. An alternative family of DG limiters which serves at the same time as a new PDE-based limiter, as well as a troubled cells indicator, was introduced by Dumbser et al. [13].

More recently, a particularly simple and compact WENO limiter was developed by Zhong and Shu [35] for RKDG schemes, and then was generalized to the unstructured mesh in [37]. This simple WENO limiter utilizes fully the advantage of DG schemes in that a complete polynomial is available in each cell without the need of reconstruction. The major advantages of this simple WENO limiter include the compactness of its stencil, the simplicity in its implementation, and the freedom in choosing linear weights, which can be set arbitrarily so long as their summation is one and each of them is nonnegative. However, it was observed in [35] that the limiter might not be robust enough for problems containing very strong shocks or low pressure problem, especially for higher order polynomials, for example the blast wave problems [30, 34] and the double rarefaction wave problem [22]. In order to overcome this difficulty, without compromising the advantages of compact stencil and simplicity of linear weights, we present a modification of the limiter in the step of preprocessing the polynomials in the immediate neighboring