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An Essential Extension of the Finite-Energy Condition for Extended Runge-Kutta-Nyström Integrators when Applied to Nonlinear Wave Equations

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Abstract. This paper is devoted to an extension of the finite-energy condition for extended Runge-Kutta-Nyström (ERKN) integrators and applications to nonlinear wave equations. We begin with an error analysis for the integrators for multi-frequency highly oscillatory systems $y''+My=f(y)$, where M is positive semi-definite, $\|M\|\gg\|\frac{\partial f}{\partial y}\|$, and $\|M\|\gg 1$. The highly oscillatory system is due to the semi-discretisation of conservative, or dissipative, nonlinear wave equations. The structure of such a matrix M and initial conditions are based on particular spatial discretisations. Similarly to the error analysis for Gaustchi-type methods of order two, where a finite-energy condition bounding amplitudes of high oscillations is satisfied by the solution, a finite-energy condition for the semi-discretisation of nonlinear wave equations is introduced and analysed. These ensure that the error bound of ERKN methods is independent of $\|M\|$. Since stepsizes are not restricted by frequencies of M, large stepsizes can be employed by our ERKN integrators of arbitrary high order. Numerical experiments provided in this paper have demonstrated that our results are truly promising, and consistent with our analysis and prediction.

AMS subject classifications: 65L70, 65L05, 65L06, 65M20

Key words: Finite-energy condition, multi-frequency highly oscillatory system, error analysis, ERKN method, nonlinear wave equation.

1 Motivation

It is known that the study of numerical methods for solving highly oscillatory problems has become increasingly important in recent decades. A major source of these problems

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is from the spatial discretisation of nonlinear wave equations, such as the Klein-Gordon equation which has received a great deal of attention in both its numerical and analytical aspects. In this paper, we pay attention to an essential extension of the finite-energy condition for ERKN integrators and applications to nonlinear wave equations.

We commence with a system of multi-frequency highly oscillatory second-order differential equations

$$\begin{cases} y''+My=f(y), & t\in[t_0,T], \\ y(t_0)=y_0, & y'(t_0)=y_0', \end{cases} \tag{1.1}$$

where $M\in\mathbb{R}^{d\times d}$ is a positive semi-definite matrix (not necessarily diagonal nor symmetric, in general), $\|M\|\gg\|\frac{\partial f}{\partial y}\|$, and $\|M\|\gg 1$. This type of problem occurs in many aspects of science and engineering, among which the spatial discretisation of nonlinear wave equations by finite difference methods or spectral methods provides a large number of practical applications. In dealing with these oscillatory problems, the adapted Runge-Kutta-Nyström (ARKN) methods and ERKN integrators were respectively proposed by Franco [4] and Yang *et. al* [42] as developments of classical Runge-Kutta-Nyström (RKN) methods. As shown in the literature (see e.g. [4,6,22,40]), based on the internal stages of traditional RKN methods, the ARKN methods adopt a new form of updates given by

$$\begin{aligned} y_{n+1}&=\phi_0(V)y_n+h\phi_1(V)y_n'+h^2\sum_{i=1}^{s}\bar{B}_i(V)f(Y_i), \\ y_{n+1}'&=-hM\phi_1(V)y_n+\phi_0(V)y_n'+h\sum_{i=1}^{s}B_i(V)f(Y_i), \end{aligned}$$

where ϕ_0, ϕ_1, $\bar{B}_i$ and B_i are matrix-valued functions, whereas, totally differently from the ARKN methods, in light of the variation-of-constants formula for (1.1), the ERKN methods not only adopt a new form of updates, but also adopt a new form of internal stages given by

$$Y_i=\phi_0(C_i^2V)y_n+C_ih\phi_1(C_i^2V)y_n'+h^2\sum_{j=1}^{s}A_{ij}(V)f(Y_j),$$

to achieve a high level of harmony with the oscillatory structure of the problem (1.1). The well-known examples of explicit ERKN integrators are Gautschi-type methods of order two [7–11,13]. As we will show by (2.10) in Section 2, the Gautschi-type method can be displayed by a Butcher tableau, which is just in the form of ERKN methods. From this observation, ERKN integrators also can be thought of as generalized Gautschi-type methods.

Another type of numerical method for solving the oscillatory problem is the exponentially (or functionally) fitted methods, such as the exponentially fitted Runge-Kutta (EFRK) method [27], the exponentially fitted Runge-Kutta-Nyström method (EFRKN) (see e.g. [5]) and the functionally-fitted energy-preserving method [19]. As stated in the