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A Third Order Adaptive ADER Scheme for One Dimensional Conservation Laws

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Abstract. We introduce a third order adaptive mesh method to arbitrary high order Godunov approach. Our adaptive mesh method consists of two parts, i.e., mesh-redistribution algorithm and solution algorithm. The mesh-redistribution algorithm is derived based on variational approach, while a new solution algorithm is developed to preserve high order numerical accuracy well. The feature of proposed Adaptive ADER scheme includes that 1). all simulations in this paper are stable for large CFL number, 2). third order convergence of the numerical solutions is successfully observed with adaptive mesh method, and 3). high resolution and non-oscillatory numerical solutions are obtained successfully when there are shocks in the solution. A variety of numerical examples show the feature well.

AMS subject classifications: 35L65, 35L67, 58J45, 35Q31, 65M08

Key words: ADER scheme, adaptive mesh method, finite volume method, WENO reconstruction, hyperbolic equations.

1 Introduction

Hyperbolic conservation laws play an important role in the computational fluid dynamics. A number of numerical methods have been developed to solve the conservation laws such as finite volume method (FVM) [12, 21, 38], discontinuous Galerkin method (DG) [26, 36], Runge-Kutta discontinuous Galerkin method (RKDG) [16, 24, 39, 40], hybridizable discontinuous Galerkin method (HDG) [6, 8], finite element method (FEM) [5, 23], spectral volume method [18, 19]. Recently, high order numerical methods attract more and more attention since their potential on delivering more efficient and accurate solutions [13,32,37]. A variety of limiting strategies have also been developed in the solution reconstruction to deliver high resolution and non-oscillatory numerical solutions, such as

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minmod [9, 20], total variation diminishing method (TVD) and total variation bounded method (TVB) [7, 10], essentially non-oscillatory scheme (ENO) and weighted essentially non-oscillatory scheme (WENO) [2, 4, 27, 29].

Among those high order numerical methods, WENO scheme is one of popular techniques to deliver non-oscillatory reconstructions. However, the optimal CFL number used in the simulation should be less than 0.5, based on numerical experience. At the same time, the WENO reconstruction should be realized three times in one temporal step if third order Runge-Kutta method is applied, for example. In order to speed up the evolution, an arbitrary high order Godunov approach (ADER) was first proposed by E.F. Toro's group [25]. A lot of further researches have been done on linear and non-linear equations, such as advection equation, Burgers equation and Euler system, on one-, two-, and three-dimension cases [31, 32]. To evaluate the numerical flux in the ADER scheme, one solves the generalized Riemann problem (GRP) [33–35]. In this procedure, Taylor expansion of the solution of GRP is performed at the interface, including a leading term and its temporal derivatives. Then all temporal derivatives are expressed in terms of spatial derivatives via Cauchy-Kowalewski procedure, which is the key idea of ADER scheme so that arbitrary high order convergence in temporal discretization can be achieved. Finally one evaluates the leading term by solving a classic Riemann problem and evaluates spatial derivatives by solving a sequence of linearized Riemann problems, respectively, at each cell interface to obtain all terms needed in the Cauchy-Kowalewski procedure and Taylor expansion, where the initial states of these Riemann problems are provided by WENO reconstruction. As was shown in their papers that the desired order convergence of ADER scheme could be achieved when linear and non-linear equations were tested. Besides, the optimal stable CFL number about 1.0 could be chosen, which sped up the evolution.

In [4], a sub-cell WENO reconstruction was proposed for ADER scheme. Numerical experiments therein successfully demonstrated the high order and non-oscillatory features of the numerical solutions. It was noted that in the simulation of Sod problem in that paper, obvious undershoot phenomenon could be observed when ADER scheme was directly used with WENO scheme proposed by Jiang and Shu (WENOJS) [29]. This phenomenon happened when the discontinuous initial condition was resolved exactly by the mesh grids, i.e., when there was a grid point appearing at the discontinuity point. Even worse, this undershoot phenomenon could not be improved by reducing the size of the mesh grids systematically. The explanation they gave in [4] is that the approximation to the leading term has much higher order accuracy than that required by the ADER scheme. We also find that this undershoot phenomenon can be significantly improved when there is an element across the discontinuity point of the initial solution. Finite volume methods propagate cell averages in the time evolution of the system. This means that an artificial smoothing for the initial condition is introduced if we let an element in the mesh cross the discontinuity point, which is negative on the reliability of the simulations. To reduce the gap between the artificially smoothed initial condition and the original one, sufficiently dense mesh grids become necessary.