Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2016-0071

Vol. **22**, No. 3, pp. 852-862
September 2017

Computation of Hyperspherical Bessel Functions

Thomas Tram*

Department of Physics and Astronomy, Aarhus University, Ny Munkegade 120, 8000 Aarhus C, Denmark.

Communicated by Jie Shen

Received 13 May 2016; Accepted (in revised version) 25 January 2017

Abstract. In this paper we present a fast and accurate numerical algorithm for the computation of hyperspherical Bessel functions of large order and real arguments. For the hyperspherical Bessel functions of closed type, no stable algorithm existed so far due to the lack of a backwards recurrence. We solved this problem by establishing a relation to Gegenbauer polynomials. All our algorithms are written in C and are publicly available at Github [`https://github.com/lesgourg/class_public`]. A Python wrapper is available upon request.

AMS subject classifications: 65Q30, 65Z05

Key words: Hyperspherical Bessel functions, recurrence relations, WKB approximation, interpolation.

1 Introduction

Hyperspherical Bessel functions are generalisations of spherical Bessel functions. They are needed for the computation of the anisotropy spectrum of the Cosmic Microwave Background (CMB) radiation for models with spatial curvature. While the differential equation can of course be integrated using ODE-solvers, no available high accuracy implementation of hyperspherical Bessel functions existed before this work. This was partly due to a problem of using backwards recurrence for hyperspherical Bessel functions of positive curvature.

2 Analytic properties of hyperspherical Bessel functions

2.1 Definition

The hyperspherical Bessel functions $\Phi_l^\nu(\chi)$ are the radial part of the eigenfunctions of the Laplacian on a 3 dimensional manifold of constant curvature. The sign of the curvature

*Corresponding author. *Email address:* `thomas.tram@phys.au.dk` (T. Tram)

is denoted by K and $\Phi_l^\nu(\chi)$ can be written as

$$\Phi_l^\nu(\chi)=\frac{u_l^\nu(\chi)}{r(\chi)}, \tag{2.1}$$

where $u_l^\nu(\chi)$ is the solution regular at $\chi=0$ of the linear second-order differential equation

$$\frac{\mathrm{d}^2u_l^\nu}{\mathrm{d}\chi^2}=\left[\frac{l(l+1)}{r(\chi)^2}-\nu^2\right]u_l^\nu(\chi). \tag{2.2}$$

The dependence on geometry is encoded in the function $r(\chi)$ given by

$$r(\chi)=\sin_K(\chi)\equiv\begin{cases}\sinh\chi, & K=-1,\\ \chi, & K=0,\\ \sin\chi, & K=1.\end{cases} \tag{2.3}$$

Note that in flat space ($K=0$), one can do the transformation $z=\nu\chi$ and multiply through by $r(\chi)^2=\chi^2$ to transform Eq. (2.2) into the Ricatti-Bessel equation, in which case $\Phi_l^\nu(\chi)=j_l(\nu\chi)$.

2.2 Recursive solutions

The solutions to Eq. (2.2) are known [1] and they can be written recursively as

$$y_l^\nu(\chi)=\begin{cases}\sinh^{l+1}\chi\left(\frac{1}{\sinh\chi}\frac{\mathrm{d}}{\mathrm{d}\chi}\right)^{l+1}(C_1\cos\nu\chi+C_2\sin\nu\chi), & K=-1,\\ \chi^{l+1}\left(\chi\frac{\mathrm{d}}{\mathrm{d}\chi}\right)^{l+1}(C_1\cos\nu\chi+C_2\sin\nu\chi), & K=0,\\ \sin^{l+1}\chi\left(\frac{1}{\sin\chi}\frac{\mathrm{d}}{\mathrm{d}\chi}\right)^{l+1}(C_1\cos\nu\chi+C_2\sin\nu\chi), & K=1.\end{cases} \tag{2.4}$$

The solution becomes regular at $x=0$ by putting $C_2=0$ which can easily be proven by induction. By letting $\chi\to-\chi$ in the solutions of Eq. (2.4), we find that $y_l^\nu(-\chi)=(-1)^l y_l^\nu(\chi)$ or equivalently

$$\Phi_l^\nu(-\chi)=(-1)^l\Phi_l^\nu(\chi). \tag{2.5}$$

Thus, the hyperspherical Bessel functions are even or odd depending on l which is well known for normal spherical Bessel functions.

2.3 Solutions in terms of Legendre functions

The hyperspherical Bessel functions for $K=\pm1$ can be expressed as Legendre functions by a change of variables [2,3]. The solutions which are regular at $x=0$ then reads

$$\Phi_l^\nu(\chi)=\begin{cases}\sqrt{\frac{\pi N_l^\nu}{2\sinh\chi}}P^{-1/2-l}_{-1/2+\mathrm{i}\nu}(\cosh\chi), & K=-1,\\ j_l(\nu\chi), & K=0,\\ \sqrt{\frac{\pi M_l^\nu}{2\sin\chi}}P^{-1/2-l}_{-1/2+\nu}(\cos\chi), & K=1,\end{cases} \tag{2.6}$$