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A Fourth Order Numerical Method for the Primitive Equations Formulated in Mean Vorticity

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Abstract. A fourth-order finite difference method is proposed and studied for the primitive equations (PEs) of large-scale atmospheric and oceanic flow based on mean vorticity formulation. Since the vertical average of the horizontal velocity field is divergence-free, we can introduce mean vorticity and mean stream function which are connected by a 2-D Poisson equation. As a result, the PEs can be reformulated such that the prognostic equation for the horizontal velocity is replaced by evolutionary equations for the mean vorticity field and the vertical derivative of the horizontal velocity. The mean vorticity equation is approximated by a compact difference scheme due to the difficulty of the mean vorticity boundary condition, while fourth-order long-stencil approximations are utilized to deal with transport type equations for computational convenience. The numerical values for the total velocity field (both horizontal and vertical) are statically determined by a discrete realization of a differential equation at each fixed horizontal point. The method is highly efficient and is capable of producing highly resolved solutions at a reasonable computational cost. The full fourth-order accuracy is checked by an example of the reformulated PEs with force terms. Additionally, numerical results of a large-scale oceanic circulation are presented.

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1 Introduction

The primitive equations (PEs) stand for fundamental governing equations for large-scale atmospheric and oceanic flow. This system is derived from the 3-D incompressible Navier-Stokes equations (NSEs) under Boussinesq assumption that density variation is neglected

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except in the buoyancy term, combined with hydrostatic approximation for the vertical momentum equation. See a detailed derivation in J. Pedlosky [24], R. Cushman [10], J. L. Lions, R. Temam and S. Wang [18–22], etc.

In the PE system, the pressure gradient, the hydrostatic balance, are coupled together with the incompressibility of the three-dimensional velocity field. In addition, there is no momentum equation for the vertical velocity since it is replaced by the hydrostatic balance. Consequently, the vertical velocity is determined by the horizontal velocity field via an integration formula of its divergence. As a result, the degree of nonlinearity of the primitive equations is even higher than that of the usual 3-D NSEs, due to lack of regularity for the vertical velocity. This nonlinearity is one of the main difficulties of the 3-D PEs, in both the PDE level and numerical analysis.

There have been numerous papers on the PDE analysis for the PEs (for example, see [2, 3, 6, 14, 16–19]). In those papers the system is proven to be well-posed. Regarding the numerical issues, some schemes based on velocity-pressure formulation were introduced and analyzed in recent articles. In [27], J. Shen and S. Wang discuss a numerical method based on a spectral Stokes solver. In [26] by R. Samelson, R. Temam, C. Wang and S. Wang, a numerical scheme in terms of the surface pressure Poisson equation formulation is proposed, and the convergence analysis of the scheme using a 3-D MAC (marker and cell) grid is established. Some relevant numerical work can also be found in [11, 30, 31], etc.

It is well-known that for 2-D NSEs, the introduction of the vorticity-stream function formulation is highly beneficial numerically and leads to the following four distinct features: (1) the vorticity and stream function are related by a kinematic Poisson equation, (2) the pressure variable is eliminated, (3) the dynamical equation is replaced by the vorticity transport equation, and (4) the velocity field is recovered by the kinematic relationship and the incompressibility is automatically enforced. We refer to [12, 13, 34] for an extensive discussion of computational methods based on local vorticity boundary conditions. In these approaches, the Neumann boundary condition for the stream function (which comes from the no-slip boundary condition for the velocity) is converted into a local vorticity boundary formula, using the kinematic relationship between the stream function and vorticity. Such an approach can be very efficiently implemented by explicit temporal discretization.

On the other hand, the development of a corresponding vorticity formulation for 3-D geophysical flow has not been as well studied. In the context of the 3-D PEs, since the leading behavior is two-dimensional by an asymptotic description of atmosphere and ocean, the above methodology can be applied in a similar, yet more tricky way. In particular, the above-mentioned four distinct features are still reflected in our vorticity formulation and numerical method as follows.

First, the averaged horizontal velocity field in vertical direction is divergence-free, namely (2.6) and (2.7) below, due to the incompressibility of the flow and the vanishing vertical velocity at the top and bottom. This allows the concept of a mean vorticity and mean stream function to be introduced so that the kinematic relationship between the