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Crouzeix-Raviart MsFEM with Bubble Functions for Diffusion and Advection-Diffusion in Perforated Media

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Abstract. The adaptation of Crouzeix-Raviart finite element in the context of multiscale finite element method (MsFEM) is studied and implemented on diffusion and advection-diffusion problems in perforated media. It is known that the approximation of boundary condition on coarse element edges when computing the multiscale basis functions critically influences the eventual accuracy of any MsFEM approaches. The weakly enforced continuity of Crouzeix-Raviart function space across element edges leads to a natural boundary condition for the multiscale basis functions which relaxes the sensitivity of our method to complex patterns of perforations. Another ingredient to our method is the application of bubble functions which is shown to be instrumental in maintaining high accuracy amid dense perforations. Additionally, the application of penalization method makes it possible to avoid complex unstructured domain and allows extensive use of simpler Cartesian meshes.

AMS subject classifications: 65N30, 34E13, 37L65, 76S05

Key words: Multiscale finite element method, Crouzeix-Raviart, porous media, bubble function.

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1 Introduction

Many important problems in modern engineering context have multiple-scale solutions e.g., transport in truly heterogeneous media like composite materials or in perforated media, or turbulence in high Reynolds number flows are some of the examples. Complete numerical analysis of these problems are difficult simply because they exhaust computational resources. In recent years, the world sees the advent of computational architectures such as parallel and GPU programming; both are shown to be advantageous to tackle resource demanding problems. Nevertheless, the size of the discrete problems remains big. In some engineering contexts, it is sometimes sufficient to predict macroscopic properties of multiscale systems. Hence it is desirable to develop an efficient computational algorithm to solve multiscale problems without being confined to solving fine scale solutions. Several methods sprung from this purpose namely, Generalized finite element methods [2], wavelet-based numerical homogenization method [10], variational multiscale method [22], various methods derived from homogenization theory [3], equation-free computations [18], heterogeneous multiscale method [23] and many others. In the context of diffusion in perforated media, some studies have been done both theoretically and numerically in [6,7,14,16,21]. For the case of advection-diffusion a method derived from heterogeneous multiscale method addressing oscillatory coefficients is studied in [9].

In this paper, we present the development of a dedicated solver for solving multiscale problems in perforated media most efficiently. We confine ourselves in dealing with only stationary diffusion and advection-diffusion problems as means to pave the way toward solving more complicated problems like Stokes. We begin by adapting the concept of multiscale finite element method (MsFEM) originally reported in [17]. The MsFEM method relies on the expansion of the solution on special basis functions which are pre-calculated by means of local simulations on a fine mesh and which model the microstructure of the problem. By contrast to sub-grid modeling methodologies, the multiscale basis functions are calculated from the actual geometry of the domain and do not depend on an (often arbitrary) analytical model of the microstructure. A study on the application of MsFEM in porous media has been done in [11], and although it could have bold significance in geo- or biosciences, they can be applied also in different contexts, e.g., pollutant dispersion in urban area [4] or on similar problems which are extremely dependent on the geometry of perforations but their full account leads to very time consuming simulations. Textbook materials on the basics of MsFEM can be found in [12].

It is understood that when constructing the multiscale basis function, the treatments of boundary condition on coarse elements greatly influence the accuracy of the method of interest. For example, in the original work of Hou and Wu, the oversampling method was introduced to provide the best approximation of the boundary condition of the multiscale basis functions which is also of high importance when dealing with non-periodic perforations. Oversampling here means that the local problem in the course element are solved on a domain larger than the element itself, but only the interior information is communicated to the coarse scale equation. This reduces the effect of wrong boundary