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# On Invariant-Preserving Finite Difference Schemes for the Camassa-Holm Equation and the Two-Component Camassa-Holm System

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**Abstract.** The purpose of this paper is to develop and test novel invariant-preserving finite difference schemes for both the Camassa-Holm (CH) equation and one of its 2-component generalizations (2CH). The considered PDEs are strongly nonlinear, admitting soliton-like peakon solutions which are characterized by a slope discontinuity at the peak in the wave shape, and therefore suitable for modeling both short wave breaking and long wave propagation phenomena. The proposed numerical schemes are shown to preserve two invariants, momentum and energy, hence numerically producing wave solutions with smaller phase error over a long time period than those generated by other conventional methods. We first apply the scheme to the CH equation and showcase the merits of considering such a scheme under a wide class of initial data. We then generalize this scheme to the 2CH equation and test this scheme under several types of initial data.

**AMS subject classifications**: 65M60, 65M12, 35Q53

**Key words**: Camassa-Holm equation, Peakon solutions, energy-preserving methods.

## 1 Introduction

This paper is concerned with the numerical approximation of a completely integrable nonlinear evolutionary partial differential equation as well as one of its two-component generalizations which often arises in various applications of shallow water wave theory. In particular, a main goal of this paper is to develop a new invariant-preserving finite difference scheme for both the Camassa-Holm (CH) equation and the two-component Camassa-Holm equation (2CH) as well as showcase the merits of using invariant-preserving methods to simulate solutions to these equations under a wide class of initial data.

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### 1.1 The CH equation

The CH equation is given by

$$m_t + um_x + 2mu_x = 0, \qquad m = u - u_{xx}, \tag{1.1}$$

which is subjected to the initial data:

$$m(x,0) = m_0(x), \tag{1.2}$$

with periodic boundary conditions. Here, $m$ is the momentum related to the fluid velocity $u$ by the one-dimensional (1-D) Helmholtz operator (see (1.1)).

Eq. (1.1) arises in a wide range of scientific applications and, for example, can be described as a bi-Hamiltonian model in the context of shallow water waves, see [4,13,14]. It can also be used to quantify growth and other changes in shape, such as those which occur in a beating heart, by providing the transformative mathematical path between two shapes (for instance, see [23, page 420]). Recalling that $m = u - u_{xx}$, the two compatible Hamiltonian descriptions of the CH equation are given by

$$m_t = -(m\partial_x + \partial_x m)\frac{\delta\mathscr{H}_1}{\delta m} = -\partial_x\left(1-\partial_x^2\right)\frac{\delta\mathscr{H}_2}{\delta m} = -\partial_x\frac{\delta\mathscr{H}_2}{\delta u}, \tag{1.3}$$

with the following conserved quantities:

$$\mathscr{H}_0 = \int_{\mathbb{R}} m(x,t)\,dx, \quad \mathscr{H}_1 = \frac{1}{2}\int_{\mathbb{R}}\left(u^2 + u_x^2\right)dx \quad \text{and} \quad \mathscr{H}_2 = \frac{1}{2}\int_{\mathbb{R}}\left(u^3 + uu_x^2\right)dx. \tag{1.4}$$

The CH equation is integrable with an infinite number of conservation laws, admits soliton-like peakon solutions, and can be viewed as a model of shallow water waves.

The CH equation (1.1) can also be written as the system of equations:

$$\begin{aligned} &u_t + uu_x + p_x = 0,\\ &p = (1-\partial_x^2)^{-1}\left(u^2 + \frac{1}{2}(u_x)^2\right), \end{aligned}$$

where $p$ is the dimensionless pressure or surface tension.

For this nonlocal conservation law with any initial data $u_0 \in H^1(\mathbb{R})$, several authors have studied the global existence of solutions, conservative or dissipative, c.f. [3, 11, 33, 36, 42]. Uniqueness is a delicate issue because in general the flow map has less regularity than usually needed to justify the uniqueness. Recently, it was proved by Bressan et al. [2] that the Cauchy problem with general initial data $u_0 \in H^1(\mathbb{R})$ has a unique conservative solution, globally in time; using a direct approach based on characteristics for the uniqueness of conservative solutions. Our goal is to compute such conservative solutions.

Simulating these peakon solutions numerically poses quite a challenge – especially if one is interested in considering a peakon-antipeakon interaction (i.e., the interaction