Commun. Comput. Phys.
doi: 10.4208/cicp.060515.161115a

Vol. **19**, No. 4, pp. 1094-1110
April 2016

A Computational Study of a Data Assimilation Algorithm for the Two-dimensional Navier-Stokes Equations

Masakazu Gesho[1], Eric Olson[2,*] and Edriss S. Titi[3,4]

[1] *Department of Chemical and Petroleum Engineering, University of Wyoming, 1000 E. University Ave, Dept. 3295, Laramie, WY 82071, USA.*

[2] *Department of Mathematics and Statistics, University of Nevada, Reno, NV 89557, USA.*

[3] *Department of Mathematics, Texas A&M University, 3368–TAMU, College Station, TX 77843, USA.*

[4] *The Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel.*

Communicated by Roger Temam

Received 6 May 2015; Accepted (in revised version) 16 November 2015

Abstract. We study the numerical performance of a continuous data assimilation (downscaling) algorithm, based on ideas from feedback control theory, in the context of the two-dimensional incompressible Navier-Stokes equations. Our model problem is to recover an unknown reference solution, asymptotically in time, by using continuous-in-time coarse-mesh nodal-point observational measurements of the velocity field of this reference solution (subsampling), as might be measured by an array of weather vane anemometers. Our calculations show that the required nodal observation density is remarkably less than what is suggested by the analytical study; and is in fact comparable to the *number of numerically determining Fourier modes*, which was reported in an earlier computational study by the authors. Thus, this method is computationally efficient and performs far better than the analytical estimates suggest.

AMS subject classifications: 35Q30, 93C20, 37C50, 76B75, 34D06

Key words: Continuous data assimilation, determining nodes, signal synchronization, two-dimensional Navier-Stokes equations, downscaling.

---

*Corresponding author. *Email addresses:* `mgesho@uwyo.edu` (M. Gesho), `ejolson@unr.edu` (E. Olson), `titi@math.tamu.edu`, `edriss.titi@weizmann.ac.il` (E. S. Titi)

1 Introduction

The goal of data assimilation is to provide a more accurate representation of the current state of a dynamical system by combining observational data with model dynamics. This allows the influences of new data to be incorporated into a numeric computation over time. Data assimilation is widely used in the climate sciences, including weather forecasting, environmental forecasting and hydrological forecasting. Additional information and historical background may be found in Kalnay [13] and references therein.

In 1969 Charney, Halem and Jastrow [5] proposed a method of continuous data assimilation in which observational measurements are directly inserted into the mathematical model as it is being integrated in time. To fix ideas, let us suppose that the evolution of u is governed by the dynamical system

$$\frac{du}{dt}=\mathcal{F}(u), \qquad u(t_0)=u_0, \tag{1.1}$$

and the observations of u are given by the time series $p(t)=Pu(t)$ for $t\in[t_0,t_*]$, where P is an orthogonal projection onto the low modes. In this context, the method proposed in [5] for approximating u from the observational data is to solve for the high modes

$$\frac{dq}{dt}=(I-P)\mathcal{F}(q+p), \qquad q(t_0)=q_0, \tag{1.2}$$

where q_0 is an arbitrarily chosen initial condition and $q+p$ represents the resulting approximation of u. Note that if $q_0=(I-P)u_0$ then $p+q=u$ for all time; however, data assimilation is applied when u_0 is not known.

Algorithm (1.2) was studied by Olson and Titi in [17] and [18] for the two-dimensional incompressible Navier-Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t}-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f, \\ \nabla\cdot u=0 \end{cases} \tag{1.3}$$

on the domain $\Omega=[0,L]^2$, equipped with periodic boundary conditions and zero spatial average with initial condition $u(x,t_0)=u_0(x)$ for $x\in[0,L]^2$. Observational measurements were represented by $P=P_h$, where P_h is the orthogonal projection onto the Fourier modes $\exp(2\pi ik\cdot x/L)$ with wave numbers $k\in\mathbf{Z}^2\setminus\{0\}$ such that $0<|k|\le L/h$. Here $\nu>0$ is the kinematic viscosity, $p(x,t)$ is the pressure and $f(x)$ is a time-independent body force with zero spatial average acting on the fluid. For simplicity, it was assumed, as we shall here, that $\nabla\cdot f=0$.

The two-dimensional incompressible Navier-Stokes equations are amenable to mathematical analysis while at the same time they posses non-linear dynamics similar to the partial differential equations that govern realistic physical phenomenon. Using the functional notation of Constantin and Foias [6], see also Temam [20] or Robinson [19], write