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Divergence-Free WENO Reconstruction-Based Finite Volume Scheme for Solving Ideal MHD Equations on Triangular Meshes

Zhiliang Xu[1,*], Dinshaw S. Balsara[2,*] and Huijing Du[3]

[1] *Department of Applied and Computational Mathematics and Statistics, University of Notre Dame, Notre Dame, IN 46556, USA.*
[2] *Department of Physics, University of Notre Dame, Notre Dame, IN 46556, USA.*
[3] *Department of Mathematics, University of California, Irvine, Irvine, CA 92697, USA.*

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Abstract. In this paper, we introduce a high-order accurate constrained transport type finite volume method to solve ideal magnetohydrodynamic equations on two-dimensional triangular meshes. A new divergence-free WENO-based reconstruction method is developed to maintain exactly divergence-free evolution of the numerical magnetic field. In this formulation, the normal component of the magnetic field at each face of a triangle is reconstructed uniquely and with the desired order of accuracy. Additionally, a new weighted flux interpolation approach is also developed to compute the z-component of the electric field at vertices of grid cells. We also present numerical examples to demonstrate the accuracy and robustness of the proposed scheme.

AMS subject classifications: 65M08, 65Z05, 85-08
Key words: Finite volume method, MHD equations, globally divergence-free.

1 Introduction

The ideal magnetohydrodynamic (MHD) equations model the dynamics of an electrically conducting fluid. Numerical solutions to MHD equations are of great importance to many applications in astrophysics and engineering. Many efforts in solving the ideal MHD equations numerically have focused on the divergence-free evolution of the magnetic field implied by the induction equation

$$\frac{\partial \mathbf{B}}{\partial t}+\nabla\times\mathbf{E}=0. \tag{1.1}$$

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[*] Corresponding author. *Email addresses:* `zxu2@nd.edu` (Z. Xu), `Dinshaw.S.Balsara.1@nd.edu` (D. S. Balsara), `huijingd@uci.edu` (H. Du)

Here $\mathbf{B}$ is the magnetic field. $\mathbf{E}$ is the electric field defined by $\mathbf{E}=-\mathbf{u}\times\mathbf{B}$ for the ideal MHD flow. $\mathbf{u}$ is the velocity, and $\mathbf{J}=\nabla\times\mathbf{B}$ is the current density. The induction equation implies that the magnetic field remains divergence-free if it is divergence-free initially. In numerical simulations, maintaining discrete divergence-free is also important. Previous studies [8,15] have shown that a divergence error on the order of numerical truncation error introduced by the numerical scheme can lead to spurious solutions and the production of negative pressures. Recent work by [47,48] has shown that positivity can be preserved in MHD simulations. Nevertheless, the magnitude of the divergence error could depend on numerical schemes. For instance, the central scheme shows smaller divergence error than the upwind-type schemes [2,3]. Recent work in [30] showed that the central scheme on overlapping cells and without using constraint transport (CT) formulation to satisfy divergence-free constraint also works for certain test problems. However, Results obtained by the central scheme without using CT formulation shows relatively large divergence error for blast wave and rotor test problems compared with these obtained by using CT formulation.

To name a few methods to ensure divergence-free evolution of the magnetic field, these include Hodge projection approach [45], Powell's source term formulation [34], locally divergence-free discontinuous Galerkin (DG) method [18,29], CT methods [4,5,7,16,19,23,36,37,44], generalized Lagrange Multiplier method [20], and many others [15,28,42].

Despite these advances, almost all previous works have been focused on rectangular meshes. The CT type divergence-free formulation on rectangular meshes has been achieved at the second-order accuracy in [6,7] and higher order of accuracy in [10]. Several problems with complex geometry require the use of triangular meshes. It is, therefore, desirable to design high-order accurate schemes possessing a globally divergence-free property for solving ideal MHD equations on unstructured meshes.

For schemes using CT type formulation on rectangular meshes, the second-order accurate representation of the magnetic field at the cell center can always be obtained by averaging the facial magnetic field. However, on the triangular meshes, this is not so straightforward, as there is no concept of arithmetic averaging of facial magnetic field to the center of the grid cells. As a result, the zone averaged magnetic field has always to be obtained via a reconstruction process on triangular meshes. This makes divergence-free formulation of schemes to solve ideal MHD equations on triangular meshes slightly more intricate than the same process on rectangular meshes.

In this paper, we introduce a divergence-free WENO reconstruction-based finite volume scheme up to the third-order accuracy for solving ideal MHD equations on two-dimensional triangular meshes. ENO and WENO finite volume schemes have been introduced in many previous works for solving scalar conservation laws as well as compressible hydrodynamical flow problems using unstructured meshes [1,21,24–26,26,41]. However, to the best of our knowledge, globally divergence-free high-order (>2) accurate finite volume schemes for solving ideal MHD equations on triangular meshes have not yet been available. To satisfy the divergence-free constraint on the magnetic field, we