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Numerical Analysis of Inverse Elasticity Problem with Signorini's Condition

Cong Zheng[1], Xiaoliang Cheng[1] and Kewei Liang[1,*]

[1] *School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, P.R. China.*

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Abstract. An optimal control problem is considered to find a stable surface traction, which minimizes the discrepancy between a given displacement field and its estimation. Firstly, the inverse elastic problem is constructed by variational inequalities, and a stable approximation of surface traction is obtained with Tikhonov regularization. Then a finite element discretization of the inverse elastic problem is analyzed. Moreover, the error estimation of the numerical solutions is deduced. Finally, a numerical algorithm is detailed and three examples in two-dimensional case illustrate the efficiency of the algorithm.

AMS subject classifications: 65M15, 65N21, 65N22, 74M15, 74S05

Key words: Optimal control, variational inequality, inverse elastic problem, finite element, error estimates, numerical simulations.

1 Introduction

Phenomena of contact between deformable bodies can be observed in daily life and many industries, such as metal forming and metal extrusion, tires with road, braking pads. The vast amounts of engineering literature have studied their mathematical models. Numerous results have been achieved in modelling, mathematical analysis and numerical simulations of contact processes [10, 14, 24]. Recent results of fundamental theories with variational and numerical analysis, including existence and uniqueness results, can be found in [6, 10, 14, 21, 22].

Since the existence, the uniqueness and the stability of the solutions are not always guaranteed [23], almost all of inverse problems are ill-posed problems. For example, the inverse heat transfer problems, which determine the temperatures or heat fluxes on the

*Corresponding author. *Email addresses:* congzheng@zju.edu.cn (C. Zheng), xiaoliangcheng@zju.edu.cn (X. Cheng), matlkw@zju.edu.cn (K. Liang)

boundary from measured internal temperatures, are very difficult to solve both numerically and analytically [2]. Some strategies have been employed for obtaining proper approximate solutions of inverse problems, such as the least squares method [5,20], regularization [12,13,19,23,25], and dynamic programming [4]. In recent years, the regularization methods are very popular, and they include iterative regularization [13,19], truncated singular value decomposition [12,13] and Tikhonov regularization [12,13,23,25].

For the inverse elastic and elasto-viscoplastic problems, preliminary work with the finite element method and a simple regularization scheme can be found in [15,16]. In [8], authors used a Laplace transform method to solve the boundary temperatures from internal displacements or stresses. In [17,18], researchers utilized a method with boundary integral equations to find plastic strains from surface displacements. Meanwhile, some numerical solutions of an inverse contact problem are given in [11,20]. However, all those papers consider the case of the body force $\boldsymbol{f}_0 = \mathbf{0}$ and the small displacement observation data $\boldsymbol{u}_d$.

In this paper, the unilateral static contact problem without friction is considered for an elastic body. The aim is to find a stable surface traction $\boldsymbol{p}$ with the given displacement field $\boldsymbol{u}_d$ in the range of the deformation theory of elasticity. We apply the idea of boundary optimal control to formulate this problem. The existence of solution has been proved in [3]. But the uniqueness of the solution can not be guaranteed. The Tikhonov regularization is applied in obtaining a stable approximate solution. Moreover, the adjoint problem of the elastic problem is introduced to construct a feasible numerical algorithm.

The paper is structured as follows. In Section 2, some notations and preliminaries are introduced. The direct problem (DP) of contact problem together with its variational formulation and the uniqueness of solution are displayed in Section 3. In Section 4, the corresponding inverse problem (IP) and the existence of its solution are showed. Meanwhile, the Tikhonov regularization is introduced into the problem to prove the uniqueness of solution, and the convergence of solution is showed. In Section 5, the discrete approximation of the IP is derived by the finite element method. We state the existence and uniqueness of the discrete solution, and derive the error estimates. A simple algorithm is applied to present several numerical simulations of two-dimensional test problem in Section 6. Finally, the summary is presented in Section 7.

2 Notation and preliminaries

In this section, we present the notation to be used throughout the paper (see [10,21,22] for more details). Let $\mathbb{N}$ be the set of all positive integers. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$, $d \in \mathbb{N}$. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$
\begin{aligned}
&\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \qquad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}}, \qquad \forall \boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^d,\\
&\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}}, \qquad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbb{S}^d.
\end{aligned}
$$