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Godunov-Type Numerical Methods for a Model of Granular Flow on Open Tables with Walls

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Abstract. We propose and analyse finite volume Godunov type methods based on discontinuous flux for a 2×2 system of non-linear partial differential equations proposed by Hadeler and Kuttler to model the dynamics of growing sandpiles generated by a vertical source on a flat bounded rectangular table. The problem considered here is the so-called partially open table problem where sand is blocked by a wall (of infinite height) on some part of the boundary of the table. The novelty here is the corresponding modification of boundary conditions for the standing and the rolling layers and generalization of the techniques of the well-balancedness proposed in [1]. Presence of walls may lead to unbounded or discontinuous surface flow density at equilibrium resulting in solutions with singularities propagating from the extreme points of the walls. A scheme has been proposed to approximate efficiently the Hamiltonians with the coefficients which can be unbounded and discontinuous. Numerical experiments are presented to illustrate that the proposed schemes detect these singularities in the equilibrium solutions efficiently and comparisons are made with the previously studied finite difference and Semi-Lagrangian approaches by Finzi Vita et al.

AMS subject classifications: 35L65, 65M12

Key words: Balance laws, discontinuous flux, granular matter, well-balanced schemes, finite volume schemes, finite difference schemes, transport rays, walls.

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1 Introduction

In this paper we continue our study on numerical methods to capture the equilibrium states for growing sandpiles. In the last decade several papers have been devoted to the study of the dynamics of granular matter since a complete and realistic description of

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many phenomena in this field is not completely available. Many mathematical models have been derived using different techniques coming from kinetic, differential equations or cellular automata theories, see [2, 3] and the references therein. This field of research has been a source of many new and challenging problems, both in theory and numerical approximations of partial differential equations, see [3–8] and references therein.

We are concerned with the evolution of a sandpile created by pouring dry sand grains, all of same size (to neglect phenomena off segregation or formation of patterns), on a flat bounded table $\Omega \subset \mathbb{R}^2$ under the effect of a time-independent non-negative vertical source $f \in L^1(\Omega)$, neglecting the external effects such as wind or stress field in the bulk of the medium. As in [1], the 2-layer (**HK**) model introduced in [9] is considered, which is an extension of the well-known BCRE model [10]. It is assumed that at a time t, the pile of sand during the evolution consists of the superposition of two non-negative different layers for any $\mathbf{x} \in \Omega$:

1. $u(\mathbf{x},t)$: the local height of the pile with the grains at rest called as the *standing* layer.
2. $v(\mathbf{x},t)$: Above the *standing* layer, there moves another layer v called as the *rolling* layer which is formed only by the grains that roll on the surface of the pile until they are captured by the standing layer.

The relations between u and v are described by the system of non-linear partial differential equations:

$$u_t = (1-|\nabla u|)v \quad \text{in} \quad \Omega \times (0,T], \tag{1.1}$$

$$v_t - \nabla.(v\nabla u) = -(1-|\nabla u|)v + f \quad \text{in} \quad \Omega \times (0,T], \tag{1.2}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}),\ v(\mathbf{x},0) = v_0(\mathbf{x}) \quad \text{in} \quad \Omega, \tag{1.3}$$

$$u(.,t) = 0 \quad \text{in} \quad \Gamma, \tag{1.4}$$

where the boundary $\partial\Omega$ can be split into two parts: Γ, an open non-empty subset of $\partial\Omega$ where the sand can fall down the table and $\Gamma_w = \partial\Omega \setminus \Gamma$ where the sand is blocked by a wall. From modelling point of view, an arbitrarily high wall can be imagined on Γ_w so that no sand can trespass this wall, while on Γ, the table is "open". If $\Gamma = \partial\Omega$, then the problem is called *totally open table problem*, otherwise it is called *partially open table problem*. In this situation, mixed boundary conditions have to be imposed on $\partial\Omega = \Gamma_w \cup \Gamma$, which were first analysed in [6] and [5] in one and higher dimensions respectively. For stability reasons, slope of u can not exceed a given constant (the critical slope), typical of the matter under consideration that is normalized to 1 and at any equilibrium configuration the local slope $|\nabla u|$ must be maximal where transport occurs (that is, where $v > 0$).

A complete mathematical theory for the existence of the solutions of (**HK**) model at finite time and at equilibrium is still not completely known and is not covered by standard existence and uniqueness results available for hyperbolic balance laws. There have been partial existence and uniqueness results on open table problem, see [1] and the reference therein for details. In case of partially open table problem, an infinite number of *Transport*